A Type Theory for Incremental Computational Complexity with Control Flow Changes (Technical Appendix)

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Structure of the Appendix

We first present the syntax and typing rules for DuCost. The remaining two sections describe the necessary definitions, lemmas and theorems for proving the soundness of abstract semantics and concrete semantics separately[1].

We use some abbreviations throughout. STS stands for "suffices to show" or "it suffices to show". TS stands for "to show" or "remains to show".

Extensions to the type system This appendix extends DuCost with refined singleton non-negative integer types: $\texttt{nat}[n]$, where n describes the value of the integer. It is eliminated with a case construct $\texttt{case}_{\texttt{N}}\ e\ \texttt{of}\ \texttt{0} \rightarrow\ e_1 \mid \texttt{succ}(x) \rightarrow\ e_2$.

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[1]We fix a mistake in the unary expression relation. All the necessary theorems take this new relation into account. In addition, r-split rule is defined only for S mode.

List of Theorems and Lemmas

Abstract Semantics

$$
\begin{array}{llll}
\text{Base Types} & B & ::= & \texttt{real} \mid \texttt{unit} \\
\text{Unannotated Types} & A & ::= & B \mid \tau_1 \times \tau_2 \mid \tau_1 + \tau_2 \mid \texttt{nat}[n] \mid \texttt{list}\,[n]^{\alpha}\ \tau \mid \\
 & & & \tau_1 \xrightarrow{\delta(\kappa)} \tau_2 \mid \forall i \overset{\delta(\kappa)}{::} S.\ \tau \mid \exists i{::}S.\ \tau \mid C \supset \tau \mid C\ \&\ \tau \\
\text{Types} & \tau & ::= & (A)^{\mu} \mid \Box(\tau) \\
\text{Modes} & \mu, \epsilon, \delta & ::= & \mathbb{S} \mid \mathbb{C} \\
\text{Sorts} & S & ::= & \mathbb{N} \mid \mathbb{R}^{+} \mid \mathbb{V} \\
\text{Index terms} & I, \kappa & ::= & i \mid \mu \mid 0 \mid I + 1 \\
 & & & I_1 + I_2 \mid I_1 - I_2 \mid \frac{I_1}{I_2} \mid I_1 \cdot I_2 \mid \lceil I \rceil \mid \lfloor I \rfloor \mid \texttt{log}_2(I) \mid I_1^{I_2} \mid \\
 & & & \texttt{min}(I_1, I_2) \mid \texttt{max}(I_1, I_2) \mid \sum_{i=I_1}^{I_n} I \mid (C\ ?\ I_1\ :\ I_2) \\
\text{Constraints} & C & ::= & I_1 \doteq I_2 \mid I_1 < I_2 \mid \neg C \mid \\
\text{Constraint env.} & \Phi & ::= & \top \mid C \wedge \Phi \\
\\
\text{Sort env.} & \Delta & ::= & \emptyset \mid \Delta, i :: S \\
\text{Type env.} & \Gamma & ::= & \emptyset \mid \Gamma, x : \tau \\
\text{Primitive env.} & \Upsilon & ::= & \emptyset \mid \Upsilon, \zeta : (B_1 \cdots B_n) \xrightarrow{\kappa} B
\end{array}
$$

Figure 1: Types

$$
\begin{array}{llll}
\text{Values} & v & ::= & \texttt{r} \mid \texttt{b} \mid (v_1, v_2) \mid \texttt{inl}\ v \mid \texttt{inr}\ v \mid \texttt{nil} \mid \texttt{cons}(v_1,\ v_2) \mid \\
 & & & \texttt{fix}\ f(x).\, e \mid \Lambda.\, e \mid \texttt{pack}\ v \mid () \\
\text{Expressions} & e, f & ::= & x \mid \texttt{r} \mid \texttt{b} \mid (e_1, e_2) \mid \texttt{fst}\ e \mid \texttt{snd}\ e \mid \texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}(e, x.e_1, y.e_2) \mid \\
 & & & \texttt{nil} \mid \texttt{cons}(e_1,\ e_2) \mid (\texttt{case}_{\texttt{L}}\ e\ \texttt{of}\ \texttt{nil}\ \rightarrow\ e_1 \mid \texttt{cons}(h, tl)\ \rightarrow\ e_2) \mid \\
 & & & \texttt{0} \mid \texttt{succ}\ e \mid (\texttt{case}_{\texttt{N}}\ e\ \texttt{of}\ \texttt{0}\ \rightarrow\ e_1 \mid \texttt{succ}(x)\ \rightarrow\ e_2) \mid \\
 & & & \texttt{fix}\ f(x).\, e \mid e_1\ e_2 \mid \zeta\ e \mid \Lambda.\, e \mid e[] \mid \texttt{pack}\ e \mid \texttt{unpack}\ e\ \texttt{as}\ x\ \texttt{in}\ e' \mid \\
 & & & \texttt{let}\, x\ =\ e_1\, \texttt{in}\ e_2 \mid e_{.\texttt{c}} \mid (e : \tau, \kappa) \mid \texttt{clet}\ e_1\ \texttt{as}\ x\ \texttt{in}\ e_2 \mid ()
\end{array}
$$

Figure 2: Value and expression syntax

$\boxed{\Psi; \Delta; \Phi \vdash \Gamma\ \texttt{wf}}$

$$
\frac{\Delta \vdash \Phi\ \texttt{wf}}{\Psi; \Delta; \Phi \vdash \cdot\ \texttt{wf}}\ \mathbf{wf}\ \cdot
\qquad
\frac{\Psi; \Delta; \Phi \vdash \Gamma\ \texttt{wf} \qquad \Psi; \Delta; \Phi \vdash \tau\ \texttt{wf}}{\Psi; \Delta; \Phi \vdash (\Gamma, x : \tau)\ \texttt{wf}}\ \mathbf{wf}\ \Gamma
$$

Figure 3: Context well-formedness

$\boxed{\Delta \vdash C\ \texttt{wf}}$

$$
\frac{\Delta \vdash I_1 :: S \qquad \Delta \vdash I_2 :: S \qquad S \in \{\mathbb{N}, \mathbb{R}^{+}\}}{\Delta \vdash I_1 < I_2\ \texttt{wf}}\ \mathbf{wf}\ <
\qquad
\frac{\Delta \vdash I_1 :: S \qquad \Delta \vdash I_2 :: S \qquad S \in \{\mathbb{N}, \mathbb{R}^{+}\}}{\Delta \vdash I_1 \doteq I_2\ \texttt{wf}}\ \mathbf{wf}\ \doteq
\qquad
\frac{\Delta \vdash C\ \texttt{wf}}{\Delta \vdash \neg C\ \texttt{wf}}\ \mathbf{wf}\ \neg
$$

Figure 4: Constraint well-formedness

$\boxed{\Delta;\Phi \vdash \tau\ \mathtt{wf}}$ $\boxed{\Delta;\Phi \vdash^{\mathsf{A}} \tau\ \mathtt{wf}}$

$$\frac{\Delta;\Phi \vdash A\ \mathtt{wf} \qquad \mu = \mathbb{S} \vee \mu = \mathbb{C}}{\Delta;\Phi \vdash (A)^{\mu}\ \mathtt{wf}}\ \textbf{wf-mu} \qquad \frac{\Delta;\Phi \vdash_{\epsilon} \tau\ \mathtt{wf} \Rightarrow \Phi}{\Delta;\Phi \vdash_{\epsilon} \Box(\tau)\ \mathtt{wf} \Rightarrow \Phi}\ \textbf{wf-box} \qquad \frac{}{\Delta;\Phi \vdash^{\mathsf{A}} \mathtt{real}\ \mathtt{wf}}\ \textbf{wf-real}$$

$$\frac{\Delta;\Phi \vdash \tau_1\ \mathtt{wf} \qquad \Delta;\Phi \vdash \tau_2\ \mathtt{wf}}{\Delta;\Phi \vdash^{\mathsf{A}} \tau_1 \times \tau_2\ \mathtt{wf}}\ \textbf{wf-pair} \qquad \frac{\Delta;\Phi \vdash \tau_1\ \mathtt{wf} \qquad \Delta;\Phi \vdash \tau_2\ \mathtt{wf}}{\Delta;\Phi \vdash^{\mathsf{A}} \tau_1 + \tau_2\ \mathtt{wf}}\ \textbf{wf-sum}$$

$$\frac{\Delta;\Phi \vdash n :: \mathbb{N} \qquad \Delta;\Phi \vdash \alpha :: \mathbb{N} \qquad \Delta;\Phi \vdash \tau\ \mathtt{wf} \qquad \Delta;\Phi \models \alpha \leq n}{\Delta;\Phi \vdash^{\mathsf{A}} \mathtt{list}\,[n]^{\alpha}\ \tau\ \mathtt{wf}}\ \textbf{wf-list} \qquad \frac{\Delta;\Phi \vdash n :: \mathbb{N}}{\Delta;\Phi \vdash^{\mathsf{A}} \mathtt{nat}[n]\ \mathtt{wf}}\ \textbf{wf-nat}$$

$$\frac{\Delta;\Phi \vdash \tau_1\ \mathtt{wf} \qquad \Delta;\Phi \vdash \tau_2\ \mathtt{wf} \qquad \Delta;\Phi \vdash \kappa :: \mathbb{R}^{+} \qquad \delta = \mathbb{S} \vee \delta = \mathbb{C}}{\Delta;\Phi \vdash^{\mathsf{A}} \tau_1 \xrightarrow{\delta(\kappa)} \tau_2\ \mathtt{wf}}\ \textbf{wf-fun}$$

$$\frac{i :: S, \Delta;\Phi \vdash \tau\ \mathtt{wf} \qquad i :: S, \Delta;\Phi \vdash \kappa :: \mathbb{R}^{+} \qquad \delta = \mathbb{S} \vee \delta = \mathbb{C}}{\Delta;\Phi \vdash^{\mathsf{A}} \forall i \overset{\delta(\kappa)}{::} S.\ \tau\ \mathtt{wf}}\ \textbf{wf-}\forall \qquad \frac{i :: S, \Delta;\Phi \vdash \tau\ \mathtt{wf}}{\Delta;\Phi \vdash^{\mathsf{A}} \exists i{::}S.\ \tau\ \mathtt{wf}}\ \textbf{wf-}\exists$$

$$\frac{}{\Delta;\Phi \vdash^{\mathsf{A}} \mathtt{unit}\ \mathtt{wf}}\ \textbf{wf-unit} \qquad \frac{\Delta;\Phi \vdash C\ \mathtt{wf} \qquad \Delta;C \wedge \Phi \vdash \tau\ \mathtt{wf}}{\Delta;\Phi \vdash^{\mathsf{A}} C \supset \tau\ \mathtt{wf}}\ \textbf{wf-C}{\rightarrow}$$

$$\frac{\Delta;\Phi \vdash C\ \mathtt{wf} \qquad \Delta;C \wedge \Phi \vdash \tau\ \mathtt{wf}}{\Delta;\Phi \vdash^{\mathsf{A}} C\ \&\ \tau\ \mathtt{wf}}\ \textbf{wf-C}{\wedge}$$

Figure 5: Well-formedness of types

$\boxed{\models \delta \trianglelefteq \tau}$

$$\frac{}{\models \mathbb{S} \trianglelefteq \tau}\ \textbf{under-T} \qquad \frac{}{\models \mathbb{C} \trianglelefteq (A)^{\mathbb{C}}}\ \textbf{under-C}$$

Figure 6: Mode under type

$$\begin{array}{rcl} \epsilon \sqcup \mathbb{S} & = & \epsilon \\ \epsilon \sqcup \mathbb{C} & = & \mathbb{C} \\ \mathbb{C} \sqcup \mu & = & \mathbb{C} \\ \mathbb{S} \sqcup \mu & = & \mu \end{array}$$

Figure 7: Upper bound of typing mode and variation annotation

$$\begin{array}{rcl} \tau^{\downarrow\mathbb{S}} & = & \tau \\ ((A)^{\mu})^{\downarrow\mathbb{C}} & = & (A)^{\mathbb{C}} \\ (\Box(\tau))^{\downarrow\mathbb{C}} & = & \tau^{\downarrow\mathbb{C}} \end{array}$$

Figure 8: Force annotation on type

$\boxed{\Delta \vdash I :: S}$

$$\frac{}{\Delta \vdash \mathbb{S} :: \mathbb{V}}\ \textbf{stable} \qquad \frac{}{\Delta \vdash \mathbb{C} :: \mathbb{V}}\ \textbf{changeable} \qquad \frac{\Delta(t) = S}{\Delta \vdash t :: S}\ \textbf{inVar} \qquad \frac{}{\Delta \vdash 0 :: \mathbb{N}}\ 0$$

$$\frac{\Delta \vdash I :: \mathbb{N}}{\Delta \vdash (I+1) :: \mathbb{N}}\ \textbf{plus} \qquad \frac{\Delta \vdash I_1 :: \mathbb{N} \qquad \Delta \vdash I_2 :: \mathbb{N} \qquad \diamond \in \{\texttt{min}, \texttt{max}, +, -, *, \div, \hat{}\}}{\Delta \vdash (I_1 \diamond I_2) :: \mathbb{N}}\ \textbf{op-bin-N}$$

$$\frac{\Delta \vdash I :: \mathbb{R}^+ \qquad \circ \in \{\lfloor\ \rfloor, \lceil\ \rceil\}}{\Delta \vdash (\circ S) :: \mathbb{N}}\ \textbf{op-un-N}$$

$$\frac{\Delta \vdash \kappa_1 :: \mathbb{R}^+ \qquad \Delta \vdash \kappa_2 :: \mathbb{R}^+ \qquad \star \in \{\texttt{min}, \texttt{max}, +, -, *, /, \hat{}\}}{\Delta \vdash (\kappa_1 \star \kappa_2) :: \mathbb{R}^+}\ \textbf{op-bin-R}$$

$$\frac{\Delta \vdash \kappa :: \mathbb{R}^+ \qquad \odot \in \{\texttt{log}_2()\}}{\Delta \vdash (\odot \kappa) :: \mathbb{R}^+}\ \textbf{op-un-R}$$

$$\frac{\Delta \vdash I_1 :: \mathbb{N} \qquad \Delta \vdash I_n :: \mathbb{N} \qquad \Delta, i :: \mathbb{N} \vdash I :: S \qquad S \in \{\mathbb{N}, \mathbb{R}^+\}}{\Delta \vdash \sum_{i=I_1}^{I_n} I :: S}\ \textbf{isum}$$

$$\frac{\Delta \vdash C\ \texttt{wf} \qquad \Delta \vdash I_1 :: \mathbb{N} \qquad \Delta \vdash I_2 :: \mathbb{N} \qquad S \in \{\mathbb{N}, \mathbb{R}^+\}}{\Delta \vdash (C\ ?\ I_1\ :\ I_2) :: S}\ \textbf{icond} \qquad \frac{\Delta \vdash I :: \mathbb{N}}{\Delta \vdash I :: \mathbb{R}^+}\ \textbf{i}\sqsubseteq$$

Figure 9: Sorting rules

$\boxed{\Delta;\Phi \models^{\mathsf{A}} A_1 \sqsubseteq A_2}$ Unannotated type A_1 is a subtype of unannotated type A_2

$\boxed{\Delta;\Phi \models \tau_1 \sqsubseteq \tau_2}$ Type τ_1 is a subtype of type τ_2

The converse of the conlusion in rules marked * can be proved using other rules (given their premises).
The rules marked with (†) are also valid for *a* judgment.

$$\frac{}{\Delta;\Phi \models (\texttt{real})^{\mathbb{S}} \sqsubseteq \Box((\texttt{real})^{\mu})}\ \textbf{real} \qquad \frac{}{\Delta;\Phi \models (\texttt{unit})^{\mathbb{S}} \sqsubseteq \Box((\texttt{unit})^{\mu})}\ \textbf{unit}$$

$$\frac{\Delta;\Phi \models \tau_1' \sqsubseteq \tau_1 \qquad \Delta;\Phi \models \tau_2 \sqsubseteq \tau_2' \qquad \Delta;\Phi \models \kappa \leq \kappa'}{\Delta;\Phi \models^{\mathsf{A}} \tau_1 \xrightarrow{\delta(\kappa)} \tau_2 \sqsubseteq \tau_1' \xrightarrow{\delta(\kappa')} \tau_2'} \to 1$$

$$\frac{}{\Delta;\Phi \models \Box((\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mu}) \sqsubseteq (\Box(\tau_1) \xrightarrow{\delta(\kappa)} \Box(\tau_2))^{\mathbb{S}}} \to \Box$$

$$\frac{t :: S, \Delta;\Phi \models \tau \sqsubseteq \tau' \qquad t :: S, \Delta;\Phi \models \kappa \leq \kappa' \qquad t \notin FV(\Phi)}{\Delta;\Phi \models^{\mathsf{A}} \forall t \overset{\delta(\kappa)}{::} S.\ \tau \sqsubseteq \forall t \overset{\delta(\kappa')}{::} S.\ \tau'} \forall 1$$

$$\frac{}{\Delta;\Phi \models \Box((\forall t \overset{\delta(\kappa)}{::} S.\ \tau)^{\mu}) \equiv (\forall t \overset{\delta(\kappa)}{::} S.\ \Box(\tau))^{\mathbb{S}}} \forall\ \Box \qquad \frac{\Delta;\Phi \models \tau_1 \sqsubseteq \tau_1' \qquad \Delta;\Phi \models \tau_2 \sqsubseteq \tau_2'}{\Delta;\Phi \models^{\mathsf{A}} \tau_1 \times \tau_2 \sqsubseteq \tau_1' \times \tau_2'} \times \mathbf{1}$$

$$\frac{}{\Delta;\Phi \models (\tau_1 \times \tau_2)^{\mu} \sqsubseteq (\tau_1^{\downarrow\mu} \times \tau_2^{\downarrow\mu})^{\mu}} \times\mu \qquad \frac{}{\Delta;\Phi \models \Box((\tau_1 \times \tau_2)^{\mu}) \sqsubseteq (\Box(\tau_1) \times \Box(\tau_2))^{\mathbb{S}}} \times\Box$$

$$\frac{\Delta;\Phi \models \tau_1 \sqsubseteq \tau_1' \qquad \Delta;\Phi \models \tau_2 \sqsubseteq \tau_2'}{\Delta;\Phi \models^{\mathsf{A}} \tau_1 + \tau_2 \sqsubseteq \tau_1' + \tau_2'} + \mathbf{1} \qquad \frac{}{\Delta;\Phi \models (\tau_1 + \tau_2)^{\mu} \equiv (\tau_1^{\downarrow\mu} + \tau_2^{\downarrow\mu})^{\mu}} + \mu$$

$$\frac{}{\Delta;\Phi \models \Box((\tau_1 + \tau_2)^{\mu}) \equiv (\Box(\tau_1) + \Box(\tau_2))^{\mathbb{S}}} + \Box$$

$$\frac{\Delta;\Phi \models n \doteq n' \qquad \Delta;\Phi \models \alpha \leq \alpha' \leq n \qquad \Delta;\Phi \models \tau \sqsubseteq \tau'}{\Delta;\Phi \models^{\mathsf{A}} \texttt{list}\,[n]^{\alpha}\ \tau \sqsubseteq \texttt{list}\,[n']^{\alpha'}\ \tau'}\ \mathbf{l1}$$

$$\frac{\Delta;\Phi \models \alpha \doteq 0}{\Delta;\Phi \models^{\mathsf{A}} \texttt{list}\,[n]^{\alpha}\ \tau \equiv \texttt{list}\,[n]^{\alpha}\ \Box(\tau)}\ \mathbf{l2^*} \qquad \frac{\Delta;\Phi \models n \doteq n'}{\Delta;\Phi \models^{\mathsf{A}} \texttt{nat}[n] \sqsubseteq \texttt{nat}[n']}\ \mathbf{nat}$$

$$\frac{}{\Delta;\Phi \models (\texttt{nat}[n])^{\mathbb{S}} \sqsubseteq \Box((\texttt{nat}[n])^{\mu})}\ \mathbf{nat}\ \Box \qquad \frac{}{\Delta;\Phi \models \Box((\texttt{list}\,[n]^{\alpha}\ \tau)^{\mu}) \equiv (\texttt{list}\,[n]^{\alpha}\ \Box(\tau))^{\mathbb{S}}}\ \mathbf{l}\Box$$

$$\frac{t :: S, \Delta;\Phi \models \tau \sqsubseteq \tau' \qquad t \notin FV(\Phi)}{\Delta;\Phi \models^{\mathsf{A}} \exists t{::}S.\ \tau \sqsubseteq \exists t{::}S.\ \tau'} \exists \mathbf{1} \qquad \frac{}{\Delta;\Phi \models (\exists t{::}S.\ \tau)^{\mu} \sqsubseteq (\exists t{::}S.\ \tau^{\downarrow\mu})^{\mu}} \exists\mu$$

$$\frac{}{\Delta;\Phi \models \Box((\exists t{::}S.\ \tau)^{\mu}) \sqsubseteq (\exists t{::}S.\ \Box(\tau))^{\mathbb{S}}} \exists\Box \qquad \frac{}{\Delta;\Phi \models \Box(\tau) \sqsubseteq \tau}\ \mathbf{T} \qquad \frac{}{\Delta;\Phi \models \Box(\tau) \sqsubseteq \Box(\Box(\tau))}\mathbf{D^*}$$

$$\frac{\Delta;\Phi \models^{\mathsf{A}} A_1 \sqsubseteq A_2}{\Delta;\Phi \models (A_1)^{\mu} \sqsubseteq (A_2)^{\mu}}\ \mathbf{C} \qquad \frac{\Delta;\Phi \models \mu_1 \leq \mu_2}{\Delta;\Phi \models (A)^{\mu_1} \sqsubseteq (A)^{\mu_2}}\ \mu \qquad \frac{}{\Delta;\Phi \models \tau \sqsubseteq \tau}\mathbf{refl^*}(\dagger)$$

$$\frac{\Delta;\Phi \models \tau_1 \sqsubseteq \tau_2 \qquad \Delta;\Phi \models \tau_2 \sqsubseteq \tau_3}{\Delta;\Phi \models \tau_1 \sqsubseteq \tau_3}\mathbf{tran}(\dagger) \qquad \frac{\Delta;\Phi \wedge C \models \eta \qquad \Delta;\Phi \wedge \neg C \models \eta}{\Delta;\Phi \models \eta}\mathbf{split}(\dagger)$$

$$\frac{\Delta;\Phi \wedge C' \models C \qquad \Delta;\Phi \models \tau \sqsubseteq \tau'}{\Delta;\Phi \models C \supset \tau \sqsubseteq C' \supset \tau'}\mathbf{c\text{-}imp} \qquad \frac{\Delta;\Phi \wedge C \models C' \qquad \Delta;\Phi \models \tau \sqsubseteq \tau'}{\Delta;\Phi \models C\ \&\ \tau \sqsubseteq C'\ \&\ \tau'}\mathbf{c\text{-}and}$$

Figure 10: Subtyping rules

$\boxed{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa}$ expression e has type τ with dynamic stability κ under the mode ϵ.

$$\frac{\kappa = ((\epsilon \doteq \mathbb{C})\ ?\ c_{var}()\ :\ 0)}{\Delta;\Phi;\Gamma, x:\tau \vdash_\epsilon x : \tau \mid \kappa}\ \textbf{var} \qquad \frac{\kappa = ((\epsilon \doteq \mathbb{C})\ ?\ c_{real}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{r} : (\mathtt{real})^{\mathbb{S}} \mid \kappa}\ \textbf{real}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : \tau_1 \mid \kappa_1 \qquad \Delta;\Phi;\Gamma \vdash_\epsilon e_2 : \tau_2 \mid \kappa_2 \qquad \kappa = (\kappa_1 + \kappa_2) + ((\epsilon \doteq \mathbb{C})\ ?\ c_{pair}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon (e_1, e_2) : (\tau_1 \times \tau_2)^{\mathbb{S}} \mid \kappa}\ \textbf{pair}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : (\tau_1 \times \tau_2)^{\mu} \mid \kappa' \qquad \models \mu \trianglelefteq \tau_1 \qquad \kappa = \kappa' + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{fst}(\epsilon,\mu)\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{fst}\ e : \tau_1 \mid \kappa}\ \textbf{fst}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : (\tau_1 \times \tau_2)^{\mu} \mid \kappa' \qquad \models \mu \trianglelefteq \tau_2 \qquad \kappa = \kappa' + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{snd}(\epsilon,\mu)\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{snd}\ e : \tau_2 \mid \kappa}\ \textbf{snd}$$

$$\frac{\Delta;\Phi \vdash (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}\ \mathtt{wf} \qquad \Delta;\Phi;f : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}, x:\tau_1, \Gamma \vdash_\delta e : \tau_2 \mid \kappa' \qquad \kappa = ((\epsilon \doteq \mathbb{C})\ ?\ c_{fix}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{fix}\ f(x).\, e : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}} \mid \kappa}\ \textbf{fix1}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mu} \mid \kappa_1 \qquad \Delta;\Phi;\Gamma \vdash_\epsilon e_2 : \tau_1 \mid \kappa_2 \\ \Delta;\Phi \models \mu \trianglelefteq \tau_2 \qquad \models (\epsilon \sqcup \mu) \leq \delta \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{app}(\epsilon,\mu)\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon e_1\ e_2 : \tau_2 \mid \kappa}\ \textbf{app}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau_1 \mid \kappa' \qquad \Delta;\Phi \vdash \tau_2\ \mathtt{wf} \qquad \kappa = \kappa' + ((\epsilon \doteq \mathbb{C})\ ?\ c_{inl}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{inl}\ e : (\tau_1 + \tau_2)^{\mathbb{S}} \mid \kappa}\ \textbf{inl}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau_2 \mid \kappa' \qquad \Delta;\Phi \vdash \tau_1\ \mathtt{wf} \qquad \kappa = \kappa' + ((\epsilon \doteq \mathbb{C})\ ?\ c_{inr}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{inr}\ e : (\tau_1 + \tau_2)^{\mathbb{S}} \mid \kappa}\ \textbf{inr}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e : (\tau_1 + \tau_2)^{\mu} \mid \kappa_e \qquad \Delta;\Phi;\Gamma, x:\tau_1 \vdash_{\epsilon\sqcup\mu} e_1 : \tau \mid \kappa' \\ \Delta;\Phi;\Gamma, y:\tau_2 \vdash_{\epsilon\sqcup\mu} e_2 : \tau \mid \kappa' \qquad \Delta;\Phi \models \mu \trianglelefteq \tau \qquad \kappa = \kappa_e + \kappa' + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{case}(\epsilon,\mu)\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{case}(e, x.e_1, y.e_2) : \tau \mid \kappa}\ \textbf{case}$$

$$\frac{\kappa = ((\epsilon \doteq \mathbb{C})\ ?\ c_{zero}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{0} : (\mathtt{nat}[0])^{\mathbb{S}} \mid \kappa}\ \textbf{zero} \qquad \frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e : (\mathtt{nat}[n])^{\mathbb{S}} \mid \kappa' \\ \kappa = \kappa' + ((\epsilon \doteq \mathbb{C})\ ?\ c_{succ}()\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{succ}\ e : (\mathtt{nat}[n+1])^{\mathbb{S}} \mid \kappa}\ \textbf{succ}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e : (\mathtt{nat}[n])^{\mathbb{S}} \mid \kappa_e \\ \Delta;\Phi \wedge n \doteq 0;\Gamma \vdash_\epsilon e_1 : \tau' \mid \kappa' \qquad i :: \iota, \Delta;\Phi \wedge n \doteq i+1;x:\mathtt{nat}[i],\Gamma \vdash_\epsilon e_2 : \tau' \mid \kappa' \\ i \notin FV(\Gamma,\Phi,\tau',\kappa') \qquad \kappa = \kappa_e + \kappa' + ((\epsilon \doteq \mathbb{C})\ ?\ c_{caseN}()\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{case_N}\ e\ \mathtt{of}\ \mathtt{0} \rightarrow\ e_1 \mid \mathtt{succ}(x) \rightarrow\ e_2 : \tau' \mid \kappa}\ \textbf{caseN}$$

$$\frac{\Delta;\Phi \vdash \tau\ \mathtt{wf} \qquad \kappa = ((\epsilon \doteq \mathbb{C})\ ?\ c_{nil}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{nil} : (\mathtt{list}\,[0]^{0}\ \tau)^{\mathbb{S}} \mid \kappa}\ \textbf{nil}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : \square(\tau) \mid \kappa_1 \\ \Delta;\Phi;\Gamma \vdash_\epsilon e_2 : (\mathtt{list}\,[n]^{\alpha}\ \tau)^{\mu} \mid \kappa_2 \qquad \kappa = \kappa_1 + \kappa_2 + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{cons}()\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{cons}(e_1,\ e_2) : (\mathtt{list}\,[n+1]^{\alpha}\ \tau)^{\mu} \mid \kappa}\ \textbf{cons1}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : \tau \mid \kappa_1 \\ \Delta;\Phi;\Gamma \vdash_\epsilon e_2 : (\mathtt{list}\,[n]^{\alpha-1}\ \tau)^{\mu} \mid \kappa_2 \qquad \Delta;\Phi \models \alpha > 0 \qquad \kappa = \kappa_1 + \kappa_2 + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{cons}()\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{cons}(e_1,\ e_2) : (\mathtt{list}\,[n+1]^{\alpha}\ \tau)^{\mu} \mid \kappa}\ \textbf{cons2}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e : (\mathtt{list}\,[n]^{\alpha}\ \tau)^{\mu} \mid \kappa_e \qquad \Delta;\Phi \wedge n \doteq 0 \wedge \alpha \doteq 0;\Gamma \vdash_{\epsilon\sqcup\mu} e_1 : \tau' \mid \kappa' \\ i :: \iota, \Delta;\Phi \wedge n \doteq i+1 \wedge \alpha \leq i;h : \square(\tau), tl : (\mathtt{list}\,[i]^{\alpha}\ \tau)^{\mu}, \Gamma \vdash_{\epsilon\sqcup\mu} e_2 : \tau' \mid \kappa' \\ i :: \iota, \beta :: \iota, \Delta;\Phi \wedge n \doteq i+1 \wedge \beta \leq i \wedge \alpha \doteq \beta + 1;h : \tau, tl : (\mathtt{list}\,[i]^{\beta}\ \tau)^{\mu}, \Gamma \vdash_{\epsilon\sqcup\mu} e_2 : \tau' \mid \kappa' \\ \models \mu \trianglelefteq \tau' i, \beta \notin FV(\Gamma,\Phi,\tau',\kappa') \qquad \kappa = \kappa_e + \kappa' + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{caseL}(\epsilon,\mu)\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{case_L}\ e\ \mathtt{of}\ \mathtt{nil} \rightarrow\ e_1 \mid \mathtt{cons}(h,tl) \rightarrow\ e_2 : \tau' \mid \kappa}\ \textbf{caseL}$$

Figure 11: Expression typing rules, part 1

$\boxed{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa}$ expression e has type τ with dynamic stability κ.

The context Υ carrying types of primitive functions is omitted from all rules.

$$\frac{t :: S, \Delta;\Phi;\Gamma \vdash_\delta e : \tau \mid \kappa' \qquad \kappa = ((\epsilon \doteq \mathbb{C})\ ?\ c_{ifun}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \Lambda.\, e : (\forall t \overset{\delta(\kappa')}{::} S.\ \tau)^{\mathbb{S}} \mid \kappa}\ \forall\mathbf{I}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e : (\forall t \overset{\delta(\kappa')}{::} S.\ \tau)^{\mu} \mid \kappa_e \\ \Delta \vdash I :: S \qquad \models \mu \trianglelefteq \tau\{I/t\} \qquad \models (\epsilon \sqcup \mu) \leq \delta \qquad \kappa = \kappa_e + \kappa'\{I/t\} + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{iApp}(\epsilon,\mu)\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon e[] : \tau\{I/t\} \mid \kappa}\ \forall\mathbf{E}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau\{I/t\} \mid \kappa' \qquad \Delta \vdash I :: S \qquad \kappa = \kappa' + ((\epsilon \doteq \mathbb{C})\ ?\ c_{pack}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{pack}\ e : (\exists t{::}S.\ \tau)^{\mathbb{S}} \mid \kappa}\ \exists\mathbf{I}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e : (\exists t{::}S.\ \tau)^{\mu} \mid \kappa_e \qquad t :: S, \Delta;\Phi;x : \tau, \Gamma \vdash_{\epsilon\sqcup\mu} e' : \tau' \mid \kappa' \\ \models \mu \trianglelefteq \tau' \qquad t \notin FV(\Phi;\Gamma,\tau',\kappa') \qquad \kappa = \kappa_e + \kappa' + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{unpack}(\epsilon,\mu)\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e' : \tau' \mid \kappa}\ \exists\mathbf{E}$$

$$\frac{\begin{array}{c}\Upsilon(\zeta) = \zeta : (B_1 \cdots B_n) \xrightarrow{\kappa'} B \\ \Delta;\Phi;\Gamma \vdash_\epsilon e : (B_i)^{\mu_i} \mid \kappa_{e_i} \qquad \mu_1 \sqcup \cdots \sqcup \mu_n = \mu \qquad \kappa = (\sum_{i=1}^{n} \kappa_{e_i}) + \kappa' + c_{prim}(\epsilon, n, \mu_1, \cdots, \mu_n)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \zeta\ (e_1 \cdots e_n) : (B)^{\mu} \mid \kappa}\ \mathbf{primApp}$$

$$\frac{\begin{array}{c}\Delta;\Phi \vdash \Box((\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}})\ \mathtt{wf} \qquad \Delta;\Phi;f : \Box((\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}}), x : \tau_1, \Gamma \vdash_\delta e : \tau_2 \mid \kappa' \\ \forall x \in \Gamma \quad \Delta;\Phi \models \Gamma(x) \sqsubseteq \Box(\Gamma(x)) \qquad \kappa = ((\epsilon \doteq \mathbb{C})\ ?\ c_{fix}()\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma,\Gamma' \vdash_\epsilon \mathtt{fix}\ f(x).\, e : \Box((\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}}) \mid \kappa}\ \mathbf{fix2}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : \tau_1 \mid \kappa_1 \qquad \Delta;\Phi;x : \tau_1, \Gamma \vdash_\epsilon e_2 : \tau_2 \mid \kappa_2 \qquad \kappa = \kappa_1 + \kappa_2 + ((\epsilon \doteq \mathbb{C})\ ?\ c_{let}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{let}\, x\ =\ e_1\ \mathtt{in}\ e_2 : \tau_2 \mid \kappa}\ \mathbf{let}$$

$$\frac{\Delta;\Phi \vdash C\ \mathtt{wf} \qquad \Delta;\Phi \wedge C;\Gamma \vdash_\epsilon e : \tau \mid \kappa' \qquad \kappa' = \kappa + ((\epsilon \doteq \mathbb{C})\ ?\ c_{impl}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon e : (C \supset \tau)^{\mathbb{S}} \mid \kappa}\ \mathbf{c\text{-}impI}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : (C \supset \tau)^{\mu} \mid \kappa' \qquad \models \mu \trianglelefteq \tau \qquad \Delta;\Phi \models C \qquad \kappa' = \kappa + ((\epsilon \sqcup \mu \doteq \mathbb{C})\ ?\ c_{dot}(\epsilon,\mu)\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon e._{\mathtt{c}} : \tau \mid \kappa}\ \mathbf{c\text{-}impE}$$

$$\frac{\Delta;\Phi \models C \qquad \Delta;\Phi \wedge C;\Gamma \vdash_\epsilon e : \tau \mid \kappa' \qquad \kappa' = \kappa + ((\epsilon \doteq \mathbb{C})\ ?\ c_{cand}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon e : (C\ \&\ \tau)^{\mathbb{S}} \mid \kappa}\ \mathbf{c\text{-}andI}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : (C\ \&\ \tau_1)^{\mu} \mid \kappa_1 \\ \Delta;\Phi \wedge C;x : \tau_1, \Gamma \vdash_{\epsilon\sqcup\mu} e_2 : \tau_2 \mid \kappa_2 \qquad \models \mu \trianglelefteq \tau_2 \qquad \kappa = \kappa_1 + \kappa_2 + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{letAs}(\epsilon,\mu)\ :\ 0)\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{clet}\ e_1\ \mathtt{as}\ x\ \mathtt{in}\ e_2 : \tau_2 \mid \kappa}\ \mathbf{c\text{-}andE}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa_e \qquad \Delta;\Phi \vdash \tau\ \mathtt{wf} \qquad FV(\tau,\kappa_e) \in \Delta}{\Delta;\Phi;\Gamma \vdash_\epsilon (e : \tau, \kappa_e) : \tau \mid \kappa_e}\ \mathbf{c\text{-}anno} \qquad \frac{\kappa = ((\epsilon \doteq \mathbb{C})\ ?\ c_{unit}()\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon () : (\mathtt{unit})^{\mathbb{S}} \mid \kappa}\ \mathbf{unit}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa' \qquad \forall x \in \Gamma \quad \Delta;\Phi \wedge C \models \Gamma(x) \sqsubseteq \Box(\Gamma(x)) \qquad \Delta;\Phi \wedge \neg C \models \kappa' \leq \kappa}{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa}\ \mathbf{r\text{-}split}$$

$$\frac{\Delta;\Phi \models \bot \qquad \Delta;\Phi \vdash \Gamma\ \mathtt{wf}}{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa}\ \mathbf{contra} \qquad \frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau' \mid \kappa' \qquad \Delta;\Phi \models \tau' \sqsubseteq \tau \qquad \Delta;\Phi \models \kappa' \leq \kappa}{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa}\ \sqsubseteq$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa' \qquad \forall x \in \Gamma \quad \Delta;\Phi \models_{\mathbb{S}} \Gamma(x) \sqsubseteq \Box(\Gamma(x)) \qquad \kappa = ((\epsilon \doteq \mathbb{S}\ ?\ 0\ :\ \kappa'))}{\Delta;\Phi;\Gamma,\Gamma' \vdash_\epsilon e : \Box(\tau) \mid \kappa}\ \mathbf{nochange}$$

Figure 12: Expression typing rules, part 2

$$
\begin{array}{lcl}
c_{var}() & = & 1 \\
c_{real}() & = & 1 \\
c_{pair}() & = & 1 \\
c_{fst}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 2\ :\ 0) \\
c_{fst}(\epsilon, \mathbb{C}) & = & 2 \\
c_{snd}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 2\ :\ 0) \\
c_{snd}(\epsilon, \mathbb{C}) & = & 2 \\
c_{fix}() & = & 1 \\
c_{app}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 2\ :\ 0) \\
c_{app}(\epsilon, \mathbb{C}) & = & 2 + (\epsilon = \mathbb{C}\ ?\ 2\ :\ 0) \\
c_{inl}() & = & 1 \\
c_{inr}() & = & 1 \\
c_{case}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 2\ :\ 0) \\
c_{case}(\epsilon, \mathbb{C}) & = & 2 \\
c_{prim}(\epsilon, n, \mathbb{S}, \ldots, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ n + 2\ :\ 0) \\
c_{prim}(\epsilon, n, \mu_1, \ldots, \mu_n) & = & n + 1 + (\epsilon = \mathbb{C}\ ?\ n\ :\ 0) \\
c_{iApp}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 2\ :\ 0) \\
c_{iApp}(\epsilon, \mathbb{C}) & = & 2 \\
c_{pack}() & = & 1 \\
c_{unpack}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 2\ :\ 0) \\
c_{unpack}(\epsilon, \mathbb{C}) & = & 2 + (\epsilon = \mathbb{C}\ ?\ 1\ :\ 0) \\
c_{unit}() & = & 1 \\
c_{let}() & = & 1 \\
c_{impl}() & = & 1 \\
c_{dot}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 1\ :\ 0) \\
c_{dot}(\epsilon, \mathbb{C}) & = & 1 \\
c_{and}() & = & 1 \\
c_{letAs}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 1\ :\ 0) \\
c_{letAs}(\epsilon, \mathbb{C}) & = & 1 \\
c_{nil}() & = & 1 \\
c_{cons}() & = & 1 \\
c_{caseL}(\epsilon, \mathbb{S}) & = & (\epsilon = \mathbb{C}\ ?\ 3\ :\ 0) \\
c_{caseL}(\epsilon, \mathbb{C}) & = & 3 \\
c_{zero}() & = & 1 \\
c_{succ}() & = & 1 \\
c_{caseN}() & = & 2
\end{array}
$$

Figure 13: Concrete costs

$$
\begin{array}{lll}
\text{Bi-values} & \mathtt{w} ::= & \mathtt{keep(r)} \mid \mathtt{new(v, v')} \mid (\mathtt{w_1}, \mathtt{w_2}) \mid \mathtt{inl\ w} \mid \mathtt{inr\ w} \mid \mathtt{0} \mid \mathtt{succ\ w} \mid \\
 & & \mathtt{nil} \mid \mathtt{cons(w_1, w_2)} \mid \mathtt{fix\ f(x).ee} \mid \Lambda.\mathtt{ee} \mid \mathtt{pack\ w} \mid () \\
\text{Bi-expressions} & \mathtt{ee} ::= & x \mid \mathtt{keep(r)} \mid \mathtt{new(v, v')} \mid (\mathtt{ee_1}, \mathtt{ee_2}) \mid \mathtt{fst\ ee} \mid \mathtt{snd\ ee} \mid \mathtt{inl\ w} \mid \mathtt{inr\ w} \mid \\
 & & (\mathtt{case(ee, x.ee_1, y, ee_2)}) \mid \mathtt{0} \mid \mathtt{succ\ ee} \mid (\mathtt{case_N\ ee\ of\ 0} \rightarrow \mathtt{ee_1} \mid \mathtt{succ\ x} \rightarrow \mathtt{ee_2}) \mid \\
 & & \mathtt{nil} \mid \mathtt{cons(ee_1, ee_2)} \mid (\mathtt{case_L\ ee\ of\ nil} \rightarrow \mathtt{ee_1} \mid \mathtt{cons(h, tl)} \rightarrow \mathtt{ee_2}) \mid \\
 & & \mathtt{fix}\ f(x).\mathtt{ee} \mid \mathtt{ee_1\ ee_2} \mid \zeta\ \mathtt{ee} \mid \Lambda.\mathtt{ee} \mid \mathtt{ee}[] \mid \mathtt{pack\ ee} \mid \mathtt{unpack\ ee\ as\ x\ in\ ee'} \mid \\
 & & \mathtt{let}\ x = \mathtt{ee_1\ in\ ee_2} \mid \mathtt{let\ ee_1\ as}\ x\ \mathtt{in\ ee_2} \mid ()
\end{array}
$$

$$\boxed{\mathtt{stable(w)} \triangleq \mathtt{new} \notin \mathtt{w} \text{ and } \mathtt{stable(ee)} \triangleq \mathtt{new} \notin \mathtt{ee}}$$

Figure 14: Syntax of bi-values and bi-expression

$\boxed{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau \text{ and } \Delta;\Phi;\Gamma \vdash_\epsilon^\kappa \mathtt{ee} \gg \tau}$ Bi-value and bi-expression typing

$$\frac{}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{keep(r)} \gg (\mathtt{real})^{\mathbb{S}}}\ \textbf{keep-r} \qquad \frac{\Delta;\Phi;\cdot \vdash_{\mathbb{C}} \mathtt{v} : \tau \mid \kappa \qquad \Delta;\Phi;\cdot \vdash_{\mathbb{C}} \mathtt{v}' : \tau \mid \kappa' \qquad \models \mathbb{C} \trianglelefteq \tau}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{new}(v, v') \gg \tau}\ \textbf{new}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w}_1 \gg \tau_1 \qquad \Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w}_2 \gg \tau_2}{\Delta;\Phi;\Gamma \vdash_\epsilon (\mathtt{w}_1, \mathtt{w}_2) \gg (\tau_1 \times \tau_2)^{\mathbb{S}}}\ \textbf{pair} \qquad \frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau_1}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{inl\ w} \gg (\tau_1 + \tau_2)^{\mathbb{S}}}\ \textbf{inl}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau_2}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{inl\ w} \gg (\tau_1 + \tau_2)^{\mathbb{S}}}\ \textbf{inr} \qquad \frac{}{\Delta;\Phi;\Gamma \vdash_\epsilon 0 \gg (\mathtt{nat}[0])^{\mathbb{S}}}\ \textbf{zero}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg (\mathtt{nat}[\mathtt{n}])^{\mathbb{S}}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{succ\ w} \gg (\mathtt{nat}[n+1])^{\mathbb{S}}}\ \textbf{succ} \qquad \frac{}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{nil} \gg (\mathtt{list}\,[0]^0\ \tau)^{\mathbb{S}}}\ \textbf{nil}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w}_1 \gg \Box(\tau) \qquad \Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w}_2 \gg (\mathtt{list}\,[\mathtt{n}]^\alpha\ \tau)^{\mathbb{S}}}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2) \gg (\mathtt{list}\,[\mathtt{n}+1]^\alpha\ \tau)^{\mathbb{S}}}\ \textbf{cons1}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w}_1 \gg \tau \qquad \Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w}_2 \gg (\mathtt{list}\,[\mathtt{n}]^{\alpha-1}\ \tau)^{\mathbb{S}} \qquad \Delta;\Phi \models \alpha > 0}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2) \gg (\mathtt{list}\,[\mathtt{n}+1]^\alpha\ \tau)^{\mathbb{S}}}\ \textbf{cons2}$$

$$\frac{\Psi; t :: S, \Delta;\Phi;\Gamma \vdash_\delta^\kappa \mathtt{ee} \gg \tau \qquad t \notin FV(\Phi;\Gamma)}{\Delta;\Phi;\Gamma \vdash_\epsilon \Lambda.\mathtt{ee} \gg (\forall \mathtt{t} \overset{\delta(\kappa)}{::} \mathtt{S}.\ \tau)^{\mathbb{S}}}\ \textbf{Lam} \qquad \frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau\{\mathtt{I/t}\} \qquad \Delta \vdash I :: \kappa}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{pack\ w} \gg (\exists \mathtt{t}{::}\mathtt{S}.\ \tau)^{\mathbb{S}}}\ \textbf{pack}$$

$$\frac{\Delta;\Phi;x : \tau_1, f : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}, \Gamma \vdash_\delta^\kappa \mathtt{ee} \gg \tau_2}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{fix}\ f(x).\,\mathtt{ee} \gg (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}}\ \textbf{fix1}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau \qquad \forall x \in \Gamma.\ \ \Delta;\Phi \models \Gamma(x) \sqsubseteq \Box(\Gamma(x)) \qquad \mathtt{stable(w)}}{\Delta;\Phi;\Gamma,\Gamma' \vdash_\epsilon \mathtt{w} \gg \Box(\tau)}\ \textbf{nochange}$$

$$\frac{\Delta;\Phi;x : \tau_1, f : \Box((\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}), \Gamma \vdash_\delta^\kappa \mathtt{ee} \gg \tau_2 \qquad \forall x \in \Gamma.\ \ \Delta;\Phi \models \Gamma(x) \sqsubseteq \Box(\Gamma(x)) \qquad \mathtt{stable(ee)}}{\Delta;\Phi;\Gamma,\Gamma' \vdash_\epsilon \mathtt{fix}\ f(x).\mathtt{ee} \gg \Box((\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}})}\ \textbf{fix2}$$

$$\frac{\Delta;\Phi \wedge C;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg (\mathtt{C} \supset \tau)^{\mathbb{S}}}\ \textbf{c-imp} \qquad \frac{\Delta;\Phi \models C \qquad \Delta;\Phi \wedge C;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg (\mathtt{C}\ \&\ \tau)^{\mathbb{S}}}\ \textbf{c-and}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau \qquad \Delta;\Phi \models \tau \sqsubseteq \tau'}{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w} \gg \tau'}\ \sqsubseteq \qquad \frac{}{\Delta;\Phi;\Gamma \vdash_\epsilon () \gg \mathtt{unit}}\ \textbf{unit}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon \mathtt{w_i} \gg \tau_\mathtt{i} \qquad \Delta;\Phi;\overline{x_i : \tau_i}, \Gamma \vdash_\epsilon e : \tau \mid \kappa}{\Delta;\Phi;\Gamma \vdash_\epsilon^\kappa \ulcorner e \urcorner[\overline{\mathtt{w_i/x_i}}] \gg \tau}\ \textbf{exp}$$

Figure 15: Typing rules for bi-values and bi-expressions

$[\![\tau]\!]_v \subseteq$ Step index $\times$ Bi-value
$[\![\tau]\!]^\kappa_\varepsilon \subseteq$ Step index $\times$ Bi-expression

$$
\begin{array}{ll}
[\![(A)^{\mathbb{S}}]\!]_v & = [\![A]\!]_v \\
[\![(A)^{\mathbb{C}}]\!]_v & = [\![A]\!]_v \cup \{(m, \mathtt{new}(v, v')) \mid \forall k.\ (k, \mathtt{v}) \in (\!|A|\!)_v \wedge (k, \mathtt{v}') \in (\!|A|\!)_v\} \\
[\![\Box(\tau)]\!]_v & = \{(m, \mathtt{w}) \mid \mathtt{stable}(\mathtt{w}) \wedge (\mathtt{m}, \mathtt{w}) \in\ [\![\tau]\!]_{\mathtt{v}}\} \\
[\![\mathtt{unit}]\!]_v & = \{(m, ()) \mid \top\} \\
[\![\mathtt{real}]\!]_v & = \{(m, \mathtt{keep}(\mathtt{r})) \mid \top\} \\
[\![\tau_1 \times \tau_2]\!]_v & = \{(m, (\mathtt{w}_1, \mathtt{w}_2)) \mid (m, \mathtt{w}_1) \in [\![\tau_1]\!]_{\mathtt{v}} \wedge (\mathtt{m}, \mathtt{w}_2) \in [\![\tau_2]\!]_{\mathtt{v}}\} \\
[\![\tau_1 + \tau_2]\!]_v & = \{(m, \mathtt{inl}\ \mathtt{w}) \mid (\mathtt{m}, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}}\} \cup \{\mathtt{inr}\ (\mathtt{m}, \mathtt{w}) \mid (\mathtt{m}, \mathtt{w}) \in [\![\tau_2]\!]_{\mathtt{v}}\} \\
[\![\mathtt{nat}[0]]\!]_v & = \{(m, \mathtt{0}) \mid \top\} \\
[\![\mathtt{nat}[n+1]]\!]_v & = \{(m, \mathtt{succ}\ \mathtt{w}) \mid (m, \mathtt{w}) \in\ [\![\mathtt{nat}[\mathtt{n}]]\!]_{\mathtt{v}}\} \\
[\![\mathtt{list}\,[0]^0\ \tau]\!]_v & = \{(m, \mathtt{nil}) \mid \top\} \\
[\![\mathtt{list}\,[n+1]^\alpha\ \tau]\!]_v & = \{(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \mid ((\mathtt{m}, \mathtt{w}_1) \in\ [\![\tau]\!]_{\mathtt{v}} \wedge (\mathtt{m}, \mathtt{w}_2) \in\ [\![\mathtt{list}\,[\mathtt{n}]^{\alpha-1}\ \tau]\!]_{\mathtt{v}} \wedge \alpha > 0) \vee \\
& \qquad ((m, \mathtt{w}_1) \in\ [\![\Box(\tau)]\!]_{\mathtt{v}} \wedge (\mathtt{m}, \mathtt{w}_2) \in\ [\![\mathtt{list}\,[\mathtt{n}]^{\alpha}\ \tau]\!]_{\mathtt{v}})\} \\
[\![\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \tau_2]\!]_v & = \{(m, \mathtt{fix}\ f(x).\mathtt{ee}) \mid\ \forall \mathtt{j} < \mathtt{m}.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}} \Rightarrow \\
& \qquad (j, \mathtt{ee}[\mathtt{fix}\ \mathtt{f}(\mathtt{x}).\mathtt{ee}/\mathtt{f}][\mathtt{w}/\mathtt{x}]) \in [\![\tau_2]\!]^\kappa_\varepsilon\} \\
[\![\tau_1 \xrightarrow{\mathbb{C}(\eta)} \tau_2]\!]_v & = \{(m, \mathtt{fix}\ f(x).\mathtt{ee}) \mid\ (\forall \mathtt{k}.\ (\mathtt{k}, \mathtt{fix}\ \mathtt{f}(\mathtt{x}).\mathrm{L}(\mathtt{ee})) \in (\!|\tau_1 \xrightarrow{\mathbb{C}(\eta)} \tau_2|\!)_{\mathtt{v}}\ \wedge \\
& \qquad (k, \mathtt{fix}\ f(x).\mathrm{R}(\mathtt{ee})) \in (\!|\tau_1 \xrightarrow{\mathbb{C}(\eta)} \tau_2|\!)_{\mathtt{v}}) \\
& \qquad \wedge (\forall j < m.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}} \Rightarrow \\
& \qquad (j, \mathtt{ee}[\mathtt{fix}\ \mathtt{f}(\mathtt{x}).\mathtt{ee}/\mathtt{f}][\mathtt{w}/\mathtt{x}]) \in [\![\tau_2]\!]^\kappa_\varepsilon)\} \\
[\![\forall t \overset{\mathbb{S}(\kappa)}{::} S.\ \tau]\!]_v & = \{(m, \Lambda.\mathtt{ee}) \mid \forall \mathtt{I}.\ \vdash \mathtt{I} :: \mathtt{S} \Rightarrow (\mathtt{m}, \mathtt{ee}) \in [\![\tau[\mathtt{I}/\mathtt{t}]]\!]^{\kappa[\mathtt{I}/\mathtt{t}]}_\varepsilon\} \\
[\![\forall t \overset{\mathbb{C}(\kappa)}{::} S.\ \tau]\!]_v & = \{(m, \Lambda.\mathtt{ee}) \mid (\forall \mathtt{k}.\ (\mathtt{k}, \mathrm{L}(\Lambda.\mathtt{ee})) \in (\!|\forall \mathtt{t} \overset{\mathbb{C}(\kappa)}{::} \mathtt{S}.\ \tau|\!)_{\mathtt{v}} \wedge (\mathtt{k}, \mathrm{R}(\Lambda.\mathtt{ee})) \in (\!|\forall \mathtt{t} \overset{\mathbb{C}(\kappa)}{::} \mathtt{S}.\ \tau|\!)_{\mathtt{v}}) \\
& \qquad \wedge (\forall I.\ \vdash I :: S \Rightarrow (m, \mathtt{ee}) \in [\![\tau[\mathtt{I}/\mathtt{t}]]\!]^{\kappa[\mathtt{I}/\mathtt{t}]}_\varepsilon)\} \\
[\![\exists t{::}S.\ \tau]\!]_v & = \{(m, \mathtt{pack}\ \mathtt{w}) \mid \exists \mathtt{I}.\vdash \mathtt{I} :: \mathtt{S}\ \wedge\ (\mathtt{m}, \mathtt{w}) \in [\![\tau[\mathtt{I}/\mathtt{t}]]\!]_{\mathtt{v}}\} \\
[\![C \supset \tau]\!]_v & = \{(m, \mathtt{w}) \mid\ \not\models \mathtt{C}\ \vee\ (\mathtt{m}, \mathtt{w}) \in\ [\![\tau]\!]_{\mathtt{v}}\} \\
[\![C\ \&\ \tau]\!]_v & = \{(m, \mathtt{w}) \mid\ \models \mathtt{C}\ \wedge\ (\mathtt{m}, \mathtt{w}) \in\ [\![\tau]\!]_{\mathtt{v}}\}
\end{array}
$$

$$
\begin{array}{ll}
[\![\tau]\!]^\kappa_\varepsilon & = \{(m, \mathtt{ee}) \mid \forall \mathtt{v}, \mathtt{D}, \mathtt{f}.\ \ \mathrm{L}(\mathtt{ee}) \Downarrow \langle \mathtt{v}, \mathtt{D}\rangle, \mathtt{f}\ \wedge\ \mathtt{f} < \mathtt{m}\ \ \Rightarrow \exists\ \mathtt{v}', \mathtt{D}', \mathtt{w}', \mathtt{c}', \mathtt{f}'\ \text{ such that} \\
& \qquad 1.\ \langle\langle v, D\rangle, \mathtt{ee}\rangle \curvearrowright \mathtt{w}', \langle \mathtt{v}', \mathtt{D}'\rangle, \mathtt{c}' \\
& \qquad 2.\ \mathrm{R}(\mathtt{ee}) \Downarrow \langle \mathtt{v}', \mathtt{D}'\rangle, \mathtt{f}' \\
& \qquad 3.\ v' = \mathrm{R}(\mathtt{w}')\ \wedge\ \mathtt{v} = \mathrm{L}(\mathtt{w}') \\
& \qquad 4.\ c' \leq \kappa \\
& \qquad 5.\ (m - f, \mathtt{w}') \in [\![\tau]\!]_{\mathtt{v}} \\
& \}
\end{array}
$$

$$
\begin{array}{ll}
\mathcal{G}[\![\cdot]\!] & = \{(k, \emptyset)\} \\
\mathcal{G}[\![\Gamma, x : \tau]\!] & = \{(m, \theta[x \mapsto \mathtt{w}]) \mid (\mathtt{m}, \theta) \in \mathcal{G}[\![\Gamma]\!]\ \wedge (\mathtt{m}, \mathtt{w}) \in\ [\![\tau]\!]_{\mathtt{v}}\}
\end{array}
$$

Figure 16: Step-indexed interpretation of types

$(\!|\tau|\!)_v \subseteq$ Step index $\times$ Value
$(\!|\tau|\!)^\kappa_\varepsilon \subseteq$ Step index $\times$ Expression

$$
\begin{array}{ll}
(\!|(A)^\mu|\!)_v & = (\!|A|\!)_v \\
(\!|\Box(\tau)|\!)_v & = (\!|\tau|\!)_v \\
(\!|\mathtt{unit}|\!)_v & = \{(k, ()) \mid \top\} \\
(\!|\mathtt{real}|\!)_v & = \{(k, \mathtt{r}) \mid \top\} \\
(\!|\tau_1 \times \tau_2|\!)_v & = \{(k, (v_1, v_2)) \mid (k, v_1) \in (\!|\tau_1|\!)_v \wedge (k, v_2) \in (\!|\tau_2|\!)_v\} \\
(\!|\tau_1 + \tau_2|\!)_v & = \{(k, \mathtt{inl}\ v) \mid (k, v) \in (\!|\tau_1|\!)_v\} \cup \{(k, \mathtt{inr}\ v) \mid (k, v) \in (\!|\tau_2|\!)_v\} \\
(\!|\mathtt{nat}[0]|\!)_v & = \{(m, \mathtt{0}) \mid \top\} \\
(\!|\mathtt{nat}[n+1]|\!)_v & = \{(m, \mathtt{succ}\ v) \mid (m, v) \in (\!|\mathtt{nat}[n]|\!)_v\} \\
(\!|\mathtt{list}\,[0]^0\ \tau|\!)_v & = \{(k, \mathtt{nil}) \mid \top\} \\
(\!|\mathtt{list}\,[n+1]^\alpha\ \tau|\!)_v & = \{(k, \mathtt{cons}(v_1,\ v_2)) \mid (k, v_1) \in\ (\!|\tau|\!)_v \wedge \\
 & \qquad ((k, v_2) \in\ (\!|\mathtt{list}\,[n]^\alpha\ \tau|\!)_v \vee (k, v_2) \in\ (\!|\mathtt{list}\,[n]^{\alpha-1}\ \tau|\!)_v)\} \\
(\!|\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \tau_2|\!)_v & = \{(k, \mathtt{fix}\ f(x).\, e) \mid \top\} \\
(\!|\tau_1 \xrightarrow{\mathbb{C}(\eta)} \tau_2|\!)_v & = \{(k, \mathtt{fix}\ f(x).\, e) \mid\ \forall j.\ j < k.\ \forall v.\ (j, v) \in (\!|\tau_1|\!)_v \Rightarrow \\
 & \qquad (j, e[\mathtt{fix}\ f(x).\, e/f][v/x]) \in (\!|\tau_2|\!)^\eta_\varepsilon\} \\
(\!|\forall t \overset{\mathbb{S}(\kappa)}{::} S.\ \tau|\!)_v & = \{(k, \Lambda.\, e) \mid \top\} \\
(\!|\forall t \overset{\mathbb{C}(\kappa)}{::} S.\ \tau|\!)_v & = \{(k, \Lambda.\, e) \mid \forall I.\ \vdash I :: S \Rightarrow (k, e) \in (\!|\tau[I/t]|\!)^{\kappa[I/t]}_\varepsilon\} \\
(\!|\exists t{::}S.\ \tau|\!)_v & = \{(k, \mathtt{pack}\ v) \mid \exists I. \vdash I :: S\ \wedge (k, v) \in (\!|\tau[I/t]|\!)_v\} \\
(\!|C \supset \tau|\!)_v & = \{(m, v) \mid\ \not\models C\ \vee\ (m, v) \in\ (\!|\tau|\!)_v\} \\
(\!|C\ \&\ \tau|\!)_v & = \{(m, v) \mid\ \models C\ \wedge\ (m, v) \in\ (\!|\tau|\!)_v\} \\
 & \\
(\!|\tau|\!)^\eta_\varepsilon & = \{(k, e) \mid \eta < k \Rightarrow (e \Downarrow \langle v, D\rangle, f \wedge f \leq \eta\ \wedge\ (k - f, v) \in\ (\!|\tau|\!)_v)\} \\
 & \\
\mathcal{G}(\!|\cdot|\!) & = \{(k, \emptyset)\} \\
\mathcal{G}(\!|\Gamma, x : \tau|\!) & = \{(k, \mathcal{U}[x \mapsto v]) \mid (k, \mathcal{U}) \in \mathcal{G}(\!|\Gamma|\!)\ \wedge (k, v) \in\ (\!|\tau|\!)_v\}
\end{array}
$$

Figure 17: Unary step-indexed interpretation of types

$$
\begin{array}{rcl||rcl}
\mathrm{L}(\mathtt{keep}(\mathtt{r})) & = & \mathtt{r} & \mathrm{R}(\mathtt{keep}(\mathtt{r})) & = & \mathtt{r} \\
\mathrm{L}(\mathtt{new}(v, v')) & = & v & \mathrm{R}(\mathtt{new}(v, v')) & = & v' \\
 & \vdots & & & \vdots &
\end{array}
$$

Homomorphic in all other syntactic constructs

If $\mathrm{L}(\mathtt{ee}) = \mathtt{e}$ and $\mathrm{R}(\mathtt{ee}) = \mathtt{e}'$, then define $\mathtt{merge}(e, e') = \mathtt{ee}$.

Figure 18: $\mathrm{L}(\mathtt{ee})$: Left or the original expression. $\mathrm{R}(\mathtt{ee})$: Right or the modified expression.

Compound Traces	T	::=	$\langle v, T\rangle$
Traces	T	::=	$\mathtt{r} \mid (T_1, T_2) \mid \mathtt{fst}\ T \mid \mathtt{snd}\ T \mid$
			$\mathtt{inl}\ T \mid \mathtt{inr}\ T \mid \mathtt{case}_{\mathtt{inl}}(T, T_r) \mid \mathtt{case}_{\mathtt{inr}}(T, T_r) \mid$
			$\mathtt{0} \mid \mathtt{succ}\ T \mid \mathtt{case}_{\mathtt{0}}(T, T_r) \mid \mathtt{case}_{\mathtt{succ}}(T, T_r) \mid$
			$\mathtt{nil} \mid \mathtt{cons}(T_1, T_2) \mid \mathtt{case}_{\mathtt{nil}}(T, T_r) \mid \mathtt{case}_{\mathtt{cons}}(T, T_r) \mid$
			$\mathtt{fix}f(x).e \mid \mathtt{app}(T_1, T_2, T_r) \mid \mathtt{primApp}(T, \zeta) \mid$
			$\Lambda.e \mid \mathtt{iApp}(T, T_r) \mid \mathtt{pack}\ T \mid \mathtt{unpack}(T, x, T_r) \mid$
			$\mathtt{let}(x, T_1, T_2) \mid \mathtt{let}_{\mathtt{as}}(x, T, T_r) \mid ()$

Figure 19: Traces

$\boxed{e \Downarrow \langle v, T\rangle, f}$ Expression e evaluates to value v with trace T and cost f

$$\frac{}{\mathtt{v} \Downarrow \langle \mathtt{v}, \mathtt{v}\rangle, 0}\ \textbf{value} \qquad \frac{}{\mathtt{r} \Downarrow \langle \mathtt{r}, \mathtt{r}\rangle, c_{real}()}\ \textbf{r} \qquad \frac{e_1 \Downarrow T_1, f_1 \qquad e_2 \Downarrow T_2, f_2 \qquad v_i = \mathtt{V}(T_i)}{(e_1, e_2) \Downarrow \langle (v_1, v_2), (T_1, T_2)\rangle, f_1 + f_2 + c_{pair}()}\ \textbf{pair}$$

$$\frac{e \Downarrow T, f \qquad (v_1, v_2) = \mathtt{V}(T)}{\mathtt{fst}\ e \Downarrow \langle v_1, \mathtt{fst}\ T\rangle, f + c_{fst}(\mathbb{C}, \_)}\ \textbf{fst} \qquad \frac{e \Downarrow T, f \qquad (v_1, v_2) = \mathtt{V}(T)}{\mathtt{fst}\ e \Downarrow \langle v_2, \mathtt{snd}\ T\rangle, f + c_{snd}(\mathbb{C}, \_)}\ \textbf{snd}$$

$$\frac{e \Downarrow T, f \qquad v = \mathtt{V}(T)}{\mathtt{inl}\ e \Downarrow \langle \mathtt{inl}\ v, \mathtt{inl}\ T\rangle, f + c_{inl}()}\ \textbf{inl} \qquad \frac{e \Downarrow T, f \qquad v = \mathtt{V}(T)}{\mathtt{inr}\ e \Downarrow \langle \mathtt{inr}\ v, \mathtt{inr}\ T\rangle, f + c_{inr}()}\ \textbf{inr}$$

$$\frac{e \Downarrow T, f \qquad \mathtt{inl}\ v = \mathtt{V}(T) \qquad e_1[v/x] \Downarrow T_r, f_r \qquad v_r = \mathtt{V}(T_r)}{\mathtt{case}(e, x.e_1, y.e_2) \Downarrow \langle v_r, \mathtt{case}_{\mathtt{inl}}(T, T_r)\rangle, f + f_r + c_{case}(\mathbb{C}, \_)}\ \textbf{case-l}$$

$$\frac{e \Downarrow T, f \qquad \mathtt{inr}\ v = \mathtt{V}(T) \qquad e_2[v/y] \Downarrow T_r, f_r \qquad v_r = \mathtt{V}(T_r)}{\mathtt{case}(e, x.e_1, y.e_2) \Downarrow \langle v_r, \mathtt{case}_{\mathtt{inr}}(T, T_r)\rangle, f + f_r + c_{case}(\mathbb{C}, \_)}\ \textbf{case-r}$$

$$\frac{}{\mathtt{0} \Downarrow \langle \mathtt{0}, \mathtt{0}\rangle, c_{zero}()}\ \textbf{zero} \qquad \frac{e \Downarrow T, f \qquad v = \mathtt{V}(T)}{\mathtt{succ}\ e \Downarrow \langle \mathtt{succ}\ v, \mathtt{succ}\ T\rangle, f + c_{succ}()}\ \textbf{succ} \qquad \frac{}{\mathtt{nil} \Downarrow \langle \mathtt{nil}, \mathtt{nil}\rangle, c_{nil}()}\ \textbf{nil}$$

$$\frac{e_1 \Downarrow T_1, f_1 \qquad e_2 \Downarrow T_2, f_2 \qquad v_i = \mathtt{V}(T_i)}{\mathtt{cons}(e_1,\ e_2) \Downarrow \langle \mathtt{cons}(v_1,\ v_2), \mathtt{cons}(T_1, T_2)\rangle, f_1 + f_2 + c_{cons}()}\ \textbf{cons}$$

$$\frac{e \Downarrow T, f_1 \qquad e_1 \Downarrow T_1, f_2 \qquad \mathtt{nil} = \mathtt{V}(T) \qquad v_1 = \mathtt{V}(T_1)}{\mathtt{case_L}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2 \Downarrow \langle v_1, \mathtt{case}_{\mathtt{nil}}(T, T_1)\rangle, f_1 + f_2 + c_{caseL}(\mathbb{C}, \_)}\ \textbf{case-nil}$$

$$\frac{e \Downarrow T, f_1 \qquad \mathtt{cons}(v_h,\ v_{tl}) = \mathtt{V}(T) \qquad e_2[v_h/h, v_{tl}/tl] \Downarrow T_2, f_2 \qquad v_2 = \mathtt{V}(T_2)}{\mathtt{case_L}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2 \Downarrow \langle v_2, \mathtt{case}_{\mathtt{cons}}(T, T_2)\rangle, f_1 + f_2 + c_{caseL}(\mathbb{C}, \_)}\ \textbf{case-cons}$$

$$\frac{}{\mathtt{fix}\ f(x).e \Downarrow \langle \mathtt{fix}\ f(x).e, \mathtt{fix} f(x).e\rangle, c_{fix}()}\ \textbf{fix}$$

$$\frac{\begin{array}{c}e_1 \Downarrow T_1, f_1 \qquad e_2 \Downarrow T_2, f_2 \qquad \mathtt{fix}\ f(x).e = \mathtt{V}(T_1) \qquad v_2 = \mathtt{V}(T_2)\\ e[v_2/x, (\mathtt{fix}\ f(x).e)/f] \Downarrow T_r, f_r \qquad v_r = \mathtt{V}(T_r)\end{array}}{e_1\ e_2 \Downarrow \langle v_r, \mathtt{app}(T_1, T_2, T_r)\rangle, f_1 + f_2 + f_r + c_{app}(\mathbb{C}, \mathbb{S})}\ \textbf{app}$$

$$\frac{e_i \Downarrow T_i, f_i \qquad v_i = \mathtt{V}(T_i) \qquad \widehat{\zeta}(v_1 \cdots v_n) = (f_r, v_r)}{\zeta\ (e_1 \cdots e_n) \Downarrow \langle v_r, \mathtt{primApp}(T_1 \cdots T_n, \zeta)\rangle, (\sum_{i=1}^{n} f_i) + f_r + c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S})}\ \textbf{primapp}$$

$$\frac{}{\Lambda.e \Downarrow \langle \Lambda.e, \Lambda.e\rangle, c_{iFun}()}\ \textbf{Lam} \qquad \frac{\begin{array}{c}e \Downarrow T, f \qquad \Lambda.e' = \mathtt{V}(T)\\ e' \Downarrow T_r, f_r \qquad v_r = \mathtt{V}(T_r)\end{array}}{e[] \Downarrow \langle v_r, \mathtt{iApp}(T, T_r)\rangle, f + f_r + c_{iApp}(\mathbb{C}, \_)}\ \textbf{iApp}$$

$$\frac{e \Downarrow T, f \qquad v = \mathtt{V}(T)}{\mathtt{pack}\ e \Downarrow \langle \mathtt{pack}\ v, \mathtt{pack}\ T\rangle, f + c_{pack}()}\ \textbf{pack}$$

$$\frac{e \Downarrow T, f \qquad \mathtt{pack}\ v = \mathtt{V}(T) \qquad e[v/x] \Downarrow T_r, f_r \qquad v_r = \mathtt{V}(T_r)}{\mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e' \Downarrow \langle v_r, \mathtt{unpack}(T, x, T_r)\rangle, f + f_r + c_{unpack}(\mathbb{C}, \mathbb{S})}\ \textbf{unpack}$$

$$\frac{e \Downarrow T, f \qquad v = \mathtt{V}(T) \qquad e'[v/x] \Downarrow T_r, f_r \qquad v_r = \mathtt{V}(T_r)}{\mathtt{let}\ x\ =\ e\ \mathtt{in}\ e' \Downarrow \langle v_r, \mathtt{let}(x, T, T_r)\rangle, f + f_r + c_{let}()}\ \textbf{let} \qquad \frac{}{() \Downarrow \langle (), ()\rangle, c_{unit}()}\ \textbf{unit}$$

$$\frac{e \Downarrow T, f \qquad v = \mathtt{V}(T) \qquad e'[v/x] \Downarrow T_r, f_r \qquad v_r = \mathtt{V}(T_r)}{\mathtt{clet}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e' \Downarrow \langle v_r, \mathtt{let}_{\mathtt{as}}(x, T, T_r)\rangle, f + f_r + c_{letAs}(\mathbb{C}, \_)}\ \textbf{clet} \qquad \frac{e \Downarrow T, f}{e.\mathtt{c} \Downarrow T, f + c_{dot}()}\ \textbf{cexpr}$$

Figure 20: From-scratch evaluation semantics

$\boxed{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'}$ Change propagation with cost-counting

In all the remaining rules except **r-nochange**, we assume that the input $\mathtt{ee}$ satisfies $\neg\mathtt{stable}(\mathtt{ee})$.

$$\frac{\mathtt{stable}(\mathtt{ee})}{\langle\langle v, T\rangle, \mathtt{ee}\rangle \curvearrowright \ulcorner \mathtt{v} \urcorner, \langle \mathtt{v}, \mathtt{T}\rangle, 0}\ \textbf{r-nochange} \qquad \frac{}{\langle\langle \mathtt{v}, T\rangle, \mathtt{new}(\_, v')\rangle \curvearrowright \mathtt{new}(v, v'), \langle \mathtt{v}', \mathtt{v}'\rangle, 0}\ \textbf{r-new}$$

$$\frac{\langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{w}_1', \mathtt{T}_1', \mathtt{c}_1' \qquad \langle T_2, \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}_2', \mathtt{T}_2', \mathtt{c}_2' \qquad v_i' = \mathtt{V}(T_i')}{\langle\langle \_, (T_1, T_2)\rangle, (\mathtt{ee}_1, \mathtt{ee}_2)\rangle \curvearrowright (\mathtt{w}_1', \mathtt{w}_2'), \langle (v_1', v_2'), (T_1', T_2')\rangle, c_1' + c_2'}\ \textbf{r-pair}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright (\mathtt{w}_1, \mathtt{w}_2), \mathtt{T}', \mathtt{c}' \qquad (v_1', v_2') = \mathtt{V}(T')}{\langle\langle \_, \mathtt{fst}\ T\rangle, \mathtt{fst}\ \mathtt{ee}\rangle \curvearrowright \mathtt{w}_1, \langle \mathtt{v}_1', \mathtt{fst}\ \mathtt{T}'\rangle, \mathtt{c}'}\ \textbf{r-fst-S}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{new}((\mathtt{v}_1, \mathtt{v}_2), (\mathtt{v}_1', \mathtt{v}_2')), \mathtt{T}', \mathtt{c}'}{\langle\langle \_, \mathtt{fst}\ T\rangle, \mathtt{fst}\ \mathtt{ee}\rangle \curvearrowright \mathtt{new}(v_1, v_1'), \langle \mathtt{v}_1', \mathtt{fst}\ T'\rangle, c' + c_{fst}(\mathbb{S}, \mathbb{C})}\ \textbf{r-fst-C}$$

$$\frac{}{\langle\langle \mathtt{fix}\ f(x).\, e', T\rangle, \mathtt{fix}\ f(x).\mathtt{ee}\rangle \curvearrowright \mathtt{fix}\ \mathtt{f(x).ee}, \langle \mathrm{R}(\mathtt{fix\ f(x).ee}), \mathrm{R}(\mathtt{fix\ f(x).ee})\rangle, 0}\ \textbf{r-fix}$$

$$\frac{\begin{array}{c}\langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{fix\ f(x).ee}, \mathtt{T}_1', \mathtt{c}_1' \\ \langle T_2, \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}_2', \mathtt{T}_2', \mathtt{c}_2' \qquad \langle T_r, \mathtt{ee}[\mathtt{w}_2'/\mathtt{x}, (\mathtt{fix\ f(x).ee})/\mathtt{f}]\rangle \curvearrowright \mathtt{w}_r', \mathtt{T}_r', \mathtt{c}_r' \qquad v_r' = \mathtt{V}(T_r')\end{array}}{\langle\langle \_, \mathtt{app}(T_1, T_2, T_r)\rangle, \mathtt{ee}_1\ \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}_r', \langle \mathtt{v}_r', \mathtt{app}(\mathtt{T}_1', \mathtt{T}_2', \mathtt{T}_r')\rangle, \mathtt{c}_1' + \mathtt{c}_2' + \mathtt{c}_r'}\ \textbf{r-app1}$$

$$\frac{\begin{array}{c}\langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{new}(\_, \mathtt{fix\ f(x).\, e'}), \mathtt{T}_1', \mathtt{c}_1' \\ \langle T_2, \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}_2', \mathtt{T}_2', \mathtt{c}_2' \qquad \mathtt{e}'[\mathrm{R}(\mathtt{w}_2')/\mathtt{x}, (\mathtt{fix\ f(x).\, e'})/\mathtt{f}] \Downarrow \mathtt{T}_r', \mathtt{f}_r' \qquad v_r' = \mathtt{V}(T_r')\end{array}}{\langle\langle v_r, \mathtt{app}(T_1, T_2, T_r)\rangle, \mathtt{ee}_1\ \mathtt{ee}_2\rangle \curvearrowright \mathtt{new}(\mathtt{v}_r, \mathtt{v}_r'), \langle \mathtt{v}_r', \mathtt{app}(\mathtt{T}_1', \mathtt{T}_2', \mathtt{T}_r')\rangle, \mathtt{c}_1' + \mathtt{c}_2' + \mathtt{f}_r' + \mathtt{c}_{\mathtt{app}}(\mathbb{S}, \mathbb{C})}\ \textbf{r-app2}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{w}, \mathtt{T}', \mathtt{c}' \qquad v' = \mathtt{V}(T')}{\langle\langle \_, \mathtt{inl}\ T\rangle, \mathtt{inl}\ \mathtt{ee}\rangle \curvearrowright \mathtt{inl}\ \mathtt{w}, \langle \mathtt{inl}\ \mathtt{v}', \mathtt{inl}\ \mathtt{T}'\rangle, \mathtt{c}'}\ \textbf{r-inl}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{w}, \mathtt{T}', \mathtt{c}' \qquad v' = \mathtt{V}(T')}{\langle\langle \_, \mathtt{inr}\ T\rangle, \mathtt{inr}\ \mathtt{ee}\rangle \curvearrowright \mathtt{inr}\ \mathtt{w}, \langle \mathtt{inr}\ \mathtt{v}', \mathtt{inr}\ \mathtt{T}'\rangle, \mathtt{c}'}\ \textbf{r-inr}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{inl}\ \mathtt{w}, \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \mathtt{ee}_1[\mathtt{w}/\mathtt{x}]\rangle \curvearrowright \mathtt{w}_r', \mathtt{T}_r', \mathtt{c}_r' \qquad v_r' = \mathtt{V}(T_r')}{\langle\langle \_, \mathtt{case}_{\mathtt{inl}}(T, T_r)\rangle, \mathtt{case}(\mathtt{ee}, \mathtt{x}.\mathtt{ee}_1, \mathtt{y}.\mathtt{ee}_2)\rangle \curvearrowright \mathtt{w}_r', \langle \mathtt{v}_r', \mathtt{case}_{\mathtt{inl}}(\mathtt{T}', \mathtt{T}_r')\rangle, \mathtt{c}' + \mathtt{c}_r'}\ \textbf{r-case-inl1}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{new}(\_, \mathtt{inl}\ \mathtt{v}'), \mathtt{T}', \mathtt{c}' \qquad \mathrm{R}(\mathtt{ee}_1)[\mathtt{v}'/\mathtt{x}] \Downarrow \mathtt{T}_r', \mathtt{f}_r' \qquad v_r' = \mathtt{V}(T_r')}{\langle\langle v_r, \mathtt{case}_{\mathtt{inl}}(T, T_r)\rangle, \mathtt{case}(\mathtt{ee}, \mathtt{x}.\mathtt{ee}_1, \mathtt{y}.\mathtt{ee}_2)\rangle \curvearrowright \mathtt{new}(\mathtt{v}_r, \mathtt{v}_r'), \langle \mathtt{v}_r', \mathtt{case}_{\mathtt{inl}}(\mathtt{T}', \mathtt{T}_r')\rangle, \mathtt{c}' + \mathtt{f}_r' + \mathtt{c}_{\mathtt{case}}(\mathbb{S}, \mathbb{C})}\ \textbf{r-case-inl2}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{new}(\_, \mathtt{inr}\ \mathtt{v}'), \mathtt{T}', \mathtt{c}' \qquad \mathrm{R}(\mathtt{ee}_2)[\mathtt{v}'/\mathtt{y}] \Downarrow \mathtt{T}_r', \mathtt{f}_r' \qquad v_r' = \mathtt{V}(T_r')}{\langle\langle v_r, \mathtt{case}_{\mathtt{inl}}(T, T_r)\rangle, \mathtt{case}(\mathtt{ee}, \mathtt{x}.\mathtt{ee}_1, \mathtt{y}.\mathtt{ee}_2)\rangle \curvearrowright \mathtt{new}(\mathtt{v}_r, \mathtt{v}_r'), \langle \mathtt{v}_r', \mathtt{case}_{\mathtt{inr}}(\mathtt{T}', \mathtt{T}_r')\rangle, \mathtt{c}' + \mathtt{f}_r' + \mathtt{c}_{\mathtt{case}}(\mathbb{S}, \mathbb{C})}\ \textbf{r-case-inl3}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{inr}\ \mathtt{w}, \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \mathtt{ee}_2[\mathtt{w}/\mathtt{y}]\rangle \curvearrowright \mathtt{w}_r', \mathtt{T}_r', \mathtt{c}_r' \qquad v_r' = \mathtt{V}(T_r')}{\langle\langle \_, \mathtt{case}_{\mathtt{inr}}(T, T_r)\rangle, \mathtt{case}(\mathtt{ee}, \mathtt{x}.\mathtt{ee}_1, \mathtt{y}.\mathtt{ee}_2)\rangle \curvearrowright \mathtt{w}_r', \langle \mathtt{v}_r', \mathtt{case}_{\mathtt{inr}}(\mathtt{T}', \mathtt{T}_r')\rangle, \mathtt{c}' + \mathtt{c}_r'}\ \textbf{r-case-inr1}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{new}(\_, \mathtt{inl}\ \mathtt{v}'), \mathtt{T}', \mathtt{c}' \qquad \mathrm{R}(\mathtt{ee}_1)[\mathtt{v}'/\mathtt{x}] \Downarrow \mathtt{T}_r', \mathtt{f}_r' \qquad v_r' = \mathtt{V}(T_r')}{\langle\langle v_r, \mathtt{case}_{\mathtt{inr}}(T, T_r)\rangle, \mathtt{case}(\mathtt{ee}, \mathtt{x}.\mathtt{ee}_1, \mathtt{y}.\mathtt{ee}_2)\rangle \curvearrowright \mathtt{new}(\mathtt{v}_r, \mathtt{v}_r'), \langle \mathtt{v}_r', \mathtt{case}_{\mathtt{inl}}(\mathtt{T}', \mathtt{T}_r')\rangle, \mathtt{c}' + \mathtt{f}_r' + \mathtt{c}_{\mathtt{case}}(\mathbb{S}, \mathbb{C})}\ \textbf{r-case-inr2}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{new}(\_, \mathtt{inr}\ \mathtt{v}'), \mathtt{T}', \mathtt{c}' \qquad \mathrm{R}(\mathtt{ee}_2)[\mathtt{v}'/\mathtt{y}] \Downarrow \mathtt{T}_r', \mathtt{f}_r' \qquad v_r' = \mathtt{V}(T_r')}{\langle\langle v_r, \mathtt{case}_{\mathtt{inr}}(T, T_r)\rangle, \mathtt{case}(\mathtt{ee}, \mathtt{x}.\mathtt{ee}_1, \mathtt{y}.\mathtt{ee}_2)\rangle \curvearrowright \mathtt{new}(\mathtt{v}_r, \mathtt{v}_r'), \langle \mathtt{v}_r', \mathtt{case}_{\mathtt{inr}}(\mathtt{T}', \mathtt{T}_r')\rangle, \mathtt{c}' + \mathtt{f}_r' + \mathtt{c}_{\mathtt{case}}(\mathbb{S}, \mathbb{C})}\ \textbf{r-case-inr3}$$

Figure 21: Change propagation rules part 1

$\boxed{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'}$ Change propagation with cost-counting

$$\frac{\langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{w}'_1, \mathtt{T}'_1, \mathtt{c}'_1 \qquad \langle T_2, \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2 \qquad \mathtt{w}'_2 \neq \mathtt{new}(\_,\_) \qquad v'_i = \mathtt{V}(T'_i)}{\langle\langle \_, \mathtt{cons}(T_1, T_2)\rangle, \mathtt{cons}(\mathtt{ee}_1, \mathtt{ee}_2)\rangle \curvearrowright \mathtt{cons}(\mathtt{w}'_1, \mathtt{w}'_2), \langle \mathtt{cons}(\mathtt{v}'_1,\ \mathtt{v}'_2), \mathtt{cons}(\mathtt{T}'_1, \mathtt{T}'_2)\rangle, \mathtt{c}'_1 + \mathtt{c}'_2}\ \textbf{r-cons1}$$

$$\frac{\langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{w}'_1, \mathtt{T}'_1, \mathtt{c}'_1 \qquad \langle T_2, \mathtt{ee}_2\rangle \curvearrowright \mathtt{new}(\_, \mathtt{v}'_{\mathtt{tl}}), \mathtt{T}'_2, \mathtt{c}'_2 \qquad v'_i = \mathtt{V}(T'_i)}{\begin{array}{l}\langle\langle \mathtt{cons}(v_h,\ v_{tl}), \mathtt{cons}(T_1, T_2)\rangle, \mathtt{cons}(\mathtt{ee}_1,\ \mathtt{ee}_2)\rangle \curvearrowright \mathtt{new}(\mathtt{cons}(\mathtt{v_h},\ \mathtt{v_{tl}}), \mathtt{cons}(\mathtt{v}'_1,\ \mathtt{v}'_{\mathtt{tl}})),\\ \qquad\qquad \langle \mathtt{cons}(v'_1,\ v'_2), \mathtt{cons}(T'_1, T'_2)\rangle, c'_1 + c'_2\end{array}}\ \textbf{r-cons2}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{nil}, \mathtt{T}', \mathtt{c}' \qquad \langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{w}'_1, \mathtt{T}'_1, \mathtt{c}'_1 \qquad v'_1 = \mathtt{V}(T'_1)}{\langle\langle \_, \mathtt{case}_{\mathtt{nil}}(T, T_1)\rangle, \mathtt{case}_{\mathtt{L}}\ \mathtt{ee}\ \mathtt{of}\ \mathtt{nil}\ \rightarrow\ \mathtt{ee}_1 \mid \mathtt{cons}(\mathtt{h}, \mathtt{tl})\ \rightarrow\ \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}'_1, \langle \mathtt{v}'_1, \mathtt{case}_{\mathtt{nil}}(\mathtt{T}', \mathtt{T}'_1)\rangle, \mathtt{c}' + \mathtt{c}'_1}\ \textbf{r-caseL-nil}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{cons}(\mathtt{w_h}, \mathtt{w_{tl}}), \mathtt{T}', \mathtt{c}' \qquad \langle T_2, \mathtt{ee}_2[\mathtt{w_h}/\mathtt{h}, \mathtt{w_{tl}}/\mathtt{tl}]\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2 \qquad v'_2 = \mathtt{V}(T'_2)}{\langle\langle \_, \mathtt{case}_{\mathtt{cons}}(T, T_2)\rangle, \mathtt{case}_{\mathtt{L}}\ \mathtt{ee}\ \mathtt{of}\ \mathtt{nil}\ \rightarrow\ \mathtt{ee}_1 \mid \mathtt{cons}(\mathtt{h}, \mathtt{tl})\ \rightarrow\ \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}'_2, \langle \mathtt{v}'_2, \mathtt{case}_{\mathtt{cons}}(\mathtt{T}', \mathtt{T}'_2)\rangle, \mathtt{c}' + \mathtt{c}'_2}\ \textbf{r-caseL1}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{new}(\_, \mathtt{cons}(\mathtt{v}'_{\mathtt{h}}, \mathtt{v}'_{\mathtt{tl}})), \mathtt{T}', \mathtt{c}' \qquad \mathrm{R}(\mathtt{ee}_2)[\mathtt{v}'_{\mathtt{h}}/\mathtt{h}, \mathtt{v}'_{\mathtt{tl}}/\mathtt{tl}] \Downarrow v'_r, T'_2, f'_r}{\begin{array}{l}\langle\langle v_r, \mathtt{case}_{\mathtt{cons}}(T, \_)\rangle, \mathtt{case}_{\mathtt{L}}\ \mathtt{ee}\ \mathtt{of}\ \mathtt{nil}\ \rightarrow\ \mathtt{ee}_1 \mid \mathtt{cons}(\mathtt{h}, \mathtt{tl})\ \rightarrow\ \mathtt{ee}_2\rangle \curvearrowright\\ \qquad \mathtt{new}(v_r, v'_r), \langle v'_r, \mathtt{case}_{\mathtt{cons}}(T', T'_2)\rangle, c' + f'_r + c_{caseL}(\mathbb{S}, \mathbb{C})\end{array}}\ \textbf{r-caseL2}$$

$$\frac{}{\langle\langle \_, \Lambda.e'\rangle, \Lambda.\mathtt{ee}\rangle \curvearrowright \Lambda.\mathtt{ee}, \langle \Lambda.\,\mathrm{R}(\mathtt{ee}), \Lambda.\mathrm{R}(\mathtt{ee})\rangle, 0}\ \textbf{r-Lam}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \Lambda.\mathtt{ee}', \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \mathtt{ee}'\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}} \qquad v'_r = \mathtt{V}(T'_r)}{\langle\langle \_, \mathtt{iApp}(T, T_r)\rangle, \mathtt{ee}[]\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \langle \mathtt{v}'_{\mathtt{r}}, \mathtt{iApp}(\mathtt{T}', \mathtt{T}'_{\mathtt{r}})\rangle, \mathtt{c}' + \mathtt{c}'_{\mathtt{r}}}\ \textbf{r-iApp1}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{new}(\_, \Lambda.\,\mathtt{e}'), \mathtt{T}', \mathtt{c}' \qquad \mathtt{e}' \Downarrow T'_r, f'_r \qquad v'_r = \mathtt{V}(T'_r)}{\langle\langle v_r, \mathtt{iApp}(T, T_r)\rangle, \mathtt{ee}[]\rangle \curvearrowright \mathtt{new}(\mathtt{v}_{\mathtt{r}}, \mathtt{v}'_{\mathtt{r}}), \langle \mathtt{v}'_{\mathtt{r}}, \mathtt{iApp}(\mathtt{T}', \mathtt{T}'_{\mathtt{r}})\rangle, \mathtt{c}' + \mathtt{f}'_{\mathtt{r}} + \mathtt{c}_{\mathtt{iApp}}(\mathbb{S}, \mathbb{C})}\ \textbf{r-iApp2}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}' \qquad v'_r = \mathtt{V}(T')}{\langle\langle \_, \mathtt{pack}\ T\rangle, \mathtt{pack}\ \mathtt{ee}\rangle \curvearrowright \mathtt{pack}\ \mathtt{w}', \langle \mathtt{pack}\ \mathtt{v}'_{\mathtt{r}}, \mathtt{pack}\ \mathtt{T}'\rangle, \mathtt{c}'}\ \textbf{r-pack}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{pack}\ \mathtt{w}', \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \mathtt{ee}'[\mathtt{w}'/\mathtt{x}]\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}} \qquad v'_r = \mathtt{V}(T')}{\langle\langle \_, \mathtt{unpack}(T, x, T_r)\rangle, \mathtt{unpack}\ \mathtt{ee}\ \mathtt{as}\ \mathtt{x}\ \mathtt{in}\ \mathtt{ee}'\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \langle \mathtt{v}'_{\mathtt{r}}, \mathtt{unpack}(\mathtt{T}', \mathtt{x}, \mathtt{T}'_{\mathtt{r}})\rangle, \mathtt{c}' + \mathtt{c}'_{\mathtt{r}}}\ \textbf{r-unpack1}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{new}(\_, \mathtt{pack}\ \mathtt{v}'), \mathtt{T}', \mathtt{c}' \qquad \mathrm{R}(\mathtt{ee}')[\mathtt{v}'/\mathtt{x}] \Downarrow v'_r, T'_r, f'_r \qquad v'_r = \mathtt{V}(T')}{\langle\langle v_r, \mathtt{unpack}(T, x, T_r)\rangle, \mathtt{unpack}\ \mathtt{ee}\ \mathtt{as}\ \mathtt{x}\ \mathtt{in}\ \mathtt{ee}'\rangle \curvearrowright \mathtt{new}(\mathtt{v}_{\mathtt{r}}, \mathtt{v}'_{\mathtt{r}}), \langle \mathtt{v}'_{\mathtt{r}}, \mathtt{unpack}(\mathtt{T}', \mathtt{x}, \mathtt{T}'_{\mathtt{r}})\rangle, \mathtt{c}' + \mathtt{f}'_{\mathtt{r}} + \mathtt{c}_{\mathtt{unpack}}(\mathbb{S}, \mathbb{C})}\ \textbf{r-unpack2}$$

$$\frac{\langle T_i, \mathtt{ee}_{\mathtt{i}}\rangle \curvearrowright \mathtt{w}'_{\mathtt{i}}, \mathtt{T}'_{\mathtt{i}}, \mathtt{c}'_{\mathtt{i}} \qquad v'_i = \mathtt{V}(T'_i) \qquad (f'_r, v'_r) = \widehat{\zeta}(v'_1 \cdots v'_n)}{\begin{array}{l}\langle\langle v_r, \mathtt{primApp}(T_1 \cdots T_n, \zeta)\rangle, \zeta\ \mathtt{ee}_1 \cdots \mathtt{ee}_{\mathtt{n}}\rangle \curvearrowright\\ \mathtt{merge}(v_r, v'_r), \langle v'_r, \mathtt{primApp}(T'_1 \cdots T'_n, \zeta)\rangle,\ (\sum_{i=1}^{n} c'_i) + f'_r + c_{prim}(\mathbb{S}, n, \mathbb{C} \cdots \mathbb{C})\end{array}}\ \textbf{r-prim}$$

$$\frac{\langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{w}'_1, \mathtt{T}'_1, \mathtt{c}'_1 \qquad \langle T_2, \mathtt{ee}_2[\mathtt{w}'_1/x]\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2}{\langle \mathtt{let}(x, T_1, T_2), \mathtt{let}\ x\ =\ \mathtt{ee}_1\ \mathtt{in}\ \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}'_2, \mathtt{let}(\mathtt{x}, \mathtt{T}'_1, \mathtt{T}'_2), \mathtt{c}'_1 + \mathtt{c}'_2}\ \textbf{r-let}$$

$$\frac{\langle T, \mathtt{ee}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'}{\langle T., \mathtt{ee}.{}_{\mathtt{c}}\rangle \curvearrowright \mathtt{w}', \mathtt{T}'., \mathtt{c}'}\ \textbf{r-cexpr}$$

$$\frac{\langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{w}'_1, \mathtt{T}'_1, \mathtt{c}'_1 \qquad \langle T_2, \mathtt{ee}_2[\mathtt{w}'_1/x]\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2}{\langle \mathtt{let}_{\mathtt{as}}(x, T_1, T_2), \mathtt{let}\ \mathtt{ee}_1\ \mathtt{as}\ x\ \mathtt{in}\ \mathtt{ee}_2\rangle \curvearrowright \mathtt{w}'_2, \mathtt{let}_{\mathtt{as}}(\mathtt{x}, \mathtt{T}'_1, \mathtt{T}'_2), \mathtt{c}'_1 + \mathtt{c}'_2}\ \textbf{r-clet-S}$$

$$\frac{\langle T_1, \mathtt{ee}_1\rangle \curvearrowright \mathtt{new}(\_, \mathtt{v}'_1), \mathtt{T}'_1, \mathtt{c}'_1 \qquad \mathrm{R}(\mathtt{ee}_2[\mathtt{w}'_1/x]) \Downarrow T'_r, f'_r \qquad v'_r = \mathtt{V}(T'_r)}{\langle \mathtt{let}_{\mathtt{as}}(x, T_1, T_2), \mathtt{let}\ \mathtt{ee}_1\ \mathtt{as}\ x\ \mathtt{in}\ \mathtt{ee}_2\rangle \curvearrowright \mathtt{new}(v_r, v'_r), \langle v'_r, \mathtt{let}_{\mathtt{as}}(x, T'_1, T'_r)\rangle, c'_1 + f'_r + c_{cand}(\mathbb{S}, \mathbb{C})}\ \textbf{r-clet-C}$$

Figure 22: Change propagation rules, part 2

Theorems and Lemmas

Lemma 1 (Sort environment substitution)
The following hold.

1. If $\Delta \vdash I :: S$ and $\Delta, i :: S \vdash I' :: S'$, then $\Delta \vdash I'[I/i] :: S'$.
2. If $\Delta \vdash I :: S$ and $\Delta, i :: S \vdash C\ \mathtt{wf}$, then $\Delta \vdash C[I/i]\ \mathtt{wf}$.
3. If $\Delta \vdash I :: S$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$, then $\vdash \sigma I :: S$.

Proof. (1) and (2) are established by simultaneous induction on the second given derivations. (3) follows from (1). □

Lemma 2 (Bi-value projection)
The following hold.

1. If $(m, \mathtt{w}) \in [\![\mathtt{A}]\!]_{\mathtt{v}}$ then $\forall k.\ (k, \mathrm{R}(\mathtt{w})) \in (\!|\mathtt{A}|\!)_{\mathtt{v}}$ and $(k, \mathrm{L}(\mathtt{w})) \in (\!|\mathtt{A}|\!)_{\mathtt{v}}$
2. If $(m, \mathtt{w}) \in [\![\tau]\!]_{\mathtt{v}}$ then $\forall k.\ (k, \mathrm{R}(\mathtt{w})) \in (\!|\tau|\!)_{\mathtt{v}}$ and $(k, \mathrm{L}(\mathtt{w})) \in (\!|\tau|\!)_{\mathtt{v}}$

Proof. For both statements, proof is by induction on the type.

Proof of statement (1): We will only show the right projection, as the left one is symmetric.

Case $(m, \mathtt{keep(n)}) \in [\![\mathtt{real}]\!]_v$

TS: $\forall k.\ (k, \mathrm{R}(\mathtt{keep(n)})) = (k, \mathtt{n}) \in (\!|\mathtt{real}|\!)_v$. This follows immediately by definition.

Case $(m, (\mathtt{w_1}, \mathtt{w_2})) \in [\![\tau_1 \times \tau_2]\!]_v$ $(\star)$

TS: $\forall k.\ (k, \mathrm{R}((\mathtt{w_1}, \mathtt{w_2}))) \in (\!|\tau_1 \times \tau_2|\!)_v$.

STS: $\forall k.\ (k, \mathrm{R}(\mathtt{w_1})) \in (\!|\tau_1|\!)_{\mathtt{v}}$ $(\diamond)$ and $(k, \mathrm{R}(\mathtt{w_2})) \in (\!|\tau_2|\!)_{\mathtt{v}}$ $(\diamond\diamond)$

By unrolling the $(\star)$, we get $(m, \mathtt{w_1}) \in [\![\tau_1]\!]_{\mathtt{v}}$ $(\dagger)$ and $(m, \mathtt{w_2}) \in [\![\tau_2]\!]_{\mathtt{v}}$ $(\dagger\dagger)$.

By IH 2 on $(\dagger)$, we get $(\diamond)$, and by IH 2 on $(\dagger\dagger)$ we get $(\diamond\diamond)$.

Case $(m, \mathtt{w}) \in [\![\tau_1 + \tau_2]\!]_{\mathtt{v}}$

$\forall k.\ (k, \mathrm{R}(\mathtt{w})) \in (\!|\tau_1 + \tau_2|\!)_{\mathtt{v}}$ There are two cases. We only show one:

We have $(m, \mathtt{inl\ w}) \in [\![\tau_1 + \tau_2]\!]_{\mathtt{v}}$, that is $(m, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}}$ $(\dagger)$.

TS: $\forall k.\ (k, \mathrm{R}(\mathtt{inl\ w})) \in (\!|\tau_1 + \tau_2|\!)_{\mathtt{v}}$ $(\star)$.

Pick k. By unrolling $(\star)$, STS: $(k, \mathrm{R}(\mathtt{w})) \in (\!|\tau_1|\!)_{\mathtt{v}}$

By IH 2 on $(\dagger)$, we get $\forall k.\ (k, \mathrm{R}(\mathtt{w})) \in (\!|\tau_1|\!)_{\mathtt{v}}$.

By instantiating with k, we conclude.

Case $(m, \mathtt{nil}) \in [\![\mathtt{list}\,[0]^0\ \tau]\!]_v$

TS: $\forall k.\ (k, \mathrm{R}(\mathtt{nil})) \in (\!|\mathtt{list}\,[0]^0\ \tau|\!)_v$

This follows immediately.

Case $(m, \mathtt{cons}(\mathtt{w_1}, \mathtt{w_2})) \in [\![\mathtt{list}\,[\mathtt{I}+1]^\alpha\ \tau]\!]_{\mathtt{v}}$ $(\star)$

TS: $\forall k.\ (k, \mathrm{R}(\mathtt{cons}(\mathtt{w_1}, \mathtt{w_2}))) \in (\!|\mathtt{list}\,[\mathtt{I}+1]^\alpha\ \tau|\!)_{\mathtt{v}}$

Pick k.

There are two cases for unrolling the definition of ($\star$).

subcase 1: We have $(m, \mathtt{w}_1) \in [\![\tau]\!]_{\mathtt{v}}$ ($\dagger$) and $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{I}]^{\alpha-1}\ \tau]\!]_{\mathtt{v}}$ ($\dagger\dagger$)
By IH 2 on ($\dagger$), we get $\forall k.\ (k, \mathrm{R}(\mathtt{w}_1)) \in (\!|\tau|\!)_{\mathtt{v}}$ ($\diamond$)
By IH 2 on ($\dagger\dagger$), we get $\forall k.\ (k, \mathrm{R}(\mathtt{w}_2)) \in (\!|\mathtt{list}\,[\mathtt{I}]^{\alpha-1}\ \tau|\!)_{\mathtt{v}}$ ($\diamond\diamond$).
By instantiating ($\diamond$) and ($\diamond\diamond$) with the k we picked above, we get
$(k, \mathrm{R}(\mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2))) \in (\!|\mathtt{list}\,[\mathtt{I}+1]^{\alpha}\ \tau|\!)_{\mathtt{v}}$.

subcase 2: We have $(m, \mathtt{w}_1) \in [\![\Box(\tau)]\!]_{\mathtt{v}}$ ($\dagger$) and $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{I}]^{\alpha}\ \tau]\!]_{\mathtt{v}}$ ($\dagger\dagger$)
By IH 2 on ($\dagger$), we get $\forall k.\ (k, \mathrm{R}(\mathtt{w}_1)) \in (\!|\tau|\!)_{\mathtt{v}}$ ($\diamond$)
By IH 2 on ($\dagger\dagger$), we get $\forall k.\ (k, \mathrm{R}(\mathtt{w}_2)) \in (\!|\mathtt{list}\,[\mathtt{I}]^{\alpha}\ \tau|\!)_{\mathtt{v}}$ ($\diamond\diamond$).
By instantiating ($\diamond$) and ($\diamond\diamond$) with the k we picked above, we get
$(k, \mathrm{R}(\mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2))) \in (\!|\mathtt{list}\,[\mathtt{I}+1]^{\alpha}\ \tau|\!)_{\mathtt{v}}$.

Case $(m, \mathtt{fix}\ f(x).\mathtt{ee}) \in [\![\tau_1 \xrightarrow{\delta(\kappa)} \tau_2]\!]_{\mathtt{v}}$
TS: $\forall k.\ (k, \mathrm{R}(\mathtt{fix}\ f(x).\mathtt{ee})) \in (\!|\tau_1 \xrightarrow{\delta(\kappa)} \tau_2|\!)_{\mathtt{v}}$.
There are two cases.

subcase 1: $\delta = \mathbb{S}$
TS: $\forall k.\ (k, \mathrm{R}(\mathtt{fix}\ f(x).\mathtt{ee}\mathbb{S})) \in (\!|\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \tau_2|\!)_{\mathtt{v}}$
This is trivial since any function is in the $(\!|\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \tau_2|\!)_{v}$.

subcase 2: $\delta = \mathbb{C}$
We have $(m, \mathtt{fix}\ f(x).\mathtt{ee}\mathbb{C}) \in [\![\tau_1 \xrightarrow{\mathbb{C}(\kappa)} \tau_2]\!]_{\mathtt{v}}$ ($\star$)
TS: $\forall k.\ (k, \mathrm{R}(\mathtt{fix}\ f(x).\mathtt{ee}\mathbb{C})) \in (\!|\tau_1 \xrightarrow{\mathbb{C}(\kappa)} \tau_2|\!)_{\mathtt{v}}$
This immediately follows by unrolling the definition ($\star$).

Case $(m, \Lambda.\mathtt{ee}) \in [\![\forall \mathtt{t} \overset{\delta(\kappa)}{::} \mathtt{S}.\ \tau]\!]_{\mathtt{v}}$
TS: $\forall k.\ (k, \mathrm{R}(\Lambda.\mathtt{ee})) \in (\!|\forall \mathtt{t} \overset{\delta(\kappa)}{::} \mathtt{S}.\ \tau|\!)_{\mathtt{v}}$.
There are two cases.

subcase 1: $\delta = \mathbb{S}$
TS: $\forall k.\ (k, \mathrm{R}(\Lambda.\mathtt{ee}\mathbb{S})) \in (\!|\forall \mathtt{t} \overset{\mathbb{S}(\kappa)}{::} \mathtt{S}.\ \tau|\!)_{\mathtt{v}}$
This is trivial since any $\Lambda.\_\mathbb{S}$ is in the $(\!|\forall t \overset{\mathbb{S}(\kappa)}{::} S.\ \tau|\!)_{v}$.

subcase 2: $\delta = \mathbb{C}$
We have $(m, \Lambda.\mathtt{ee}\mathbb{C}) \in [\![\forall \mathtt{t} \overset{\mathbb{C}(\kappa)}{::} \mathtt{S}.\ \tau]\!]_{\mathtt{v}}$ ($\star$)
TS: $\forall k.\ (k, \mathrm{R}(\Lambda.\mathtt{ee}\mathbb{C})) \in (\!|\forall \mathtt{t} \overset{\mathbb{C}(\kappa)}{::} \mathtt{S}.\ \tau|\!)_{\mathtt{v}}$
This immediately follows by unrolling ($\star$).

Proof of statement (2):

Case $(m, \mathtt{w}) \in [\![(\mathtt{A})^{\mathbb{S}}]\!]_{\mathtt{v}} = [\![\mathtt{A}]\!]_{\mathtt{v}}$

TS: $\forall k.\ (k, \mathrm{R}(\mathtt{w})) \in (\!|(\mathtt{A})^{\mathbb{S}}|\!)_{\mathtt{v}} = (\!|\mathtt{A}|\!)_{\mathtt{v}}$.

Immediately follows by IH 1 on $(m, \mathtt{w}) \in [\![\mathtt{A}]\!]_{\mathtt{v}}$.

Case $(m, \mathtt{w}) \in [\![(\mathtt{A})^{\mathbb{C}}]\!]_{\mathtt{v}}$

TS: $\forall k.\ (k, \mathrm{R}(\mathtt{w})) \in (\!|(\mathtt{A})^{\mathbb{C}}|\!)_{\mathtt{v}} = (\!|\mathtt{A}|\!)_{\mathtt{v}}$.

There are two cases.

subcase 1: $(m, \mathtt{w}) \in [\![(\mathtt{A})^{\mathbb{C}}]\!]_{\mathtt{v}} = [\![\mathtt{A}]\!]_{\mathtt{v}}$

The proof is same as the previous case, by IH 1.

subcase 2: $(m, \mathtt{w}) \in [\![(\mathtt{A})^{\mathbb{C}}]\!]_{\mathtt{v}}$

Then by definition, we immediately get $\forall k.\ (k, \mathrm{R}(\mathtt{w})) \in (\!|(\mathtt{A})^{\mathbb{C}}|\!)_{\mathtt{v}}$.

Case $(m, \mathtt{w}) \in [\![\Box(\tau)]\!]_{\mathtt{v}}$

By unrolling the definition we know that $(m, \mathtt{w}) \in [\![\tau]\!]_{\mathtt{v}}(\spadesuit)$.

TS: $\forall k.\ (k, \mathrm{R}(\mathtt{w})) \in (\!|\Box(\tau)|\!)_{\mathtt{v}} = (\!|\tau|\!)_{\mathtt{v}}$.

Follows immediately by IH 2 on $(\spadesuit)$.

□

Lemma 3 (Downward closure)
The following hold.

1. If $(m, \mathtt{w}) \in [\![\tau]\!]_{\mathtt{v}}$ and $m' \leq m$, then $(m', \mathtt{w}) \in [\![\tau]\!]_{\mathtt{v}}$.
2. If $(m, v) \in (\!|\tau|\!)_{v}$ and $m' \leq m$, then $(m', v) \in (\!|\tau|\!)_{v}$.
3. If $(m, \mathtt{w}) \in [\![\tau]\!]^{\kappa}_{\varepsilon}$ and $m' \leq m$, then $(m', \mathtt{w}) \in [\![\tau]\!]^{\kappa}_{\varepsilon}$.
4. If $(m, e) \in (\!|\tau|\!)^{\kappa}_{\varepsilon}$ and $m' \leq m$, then $(m', e) \in (\!|\tau|\!)^{\kappa}_{\varepsilon}$.
5. If $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ and $m' \leq m$, then $(m', \theta) \in \mathcal{G}[\![\Gamma]\!]$.
6. If $(m, \mathcal{U}) \in \mathcal{G}(\!|\Gamma|\!)$ and $m' \leq m$, then $(m', \mathcal{U}) \in \mathcal{G}(\!|\Gamma|\!)$.

Proof. (1,3) and (2,4) are proved simultaneously by induction on τ. (5,6) follows from (1,2). □

Lemma 4 (Bi-value propagation)
$\langle \mathrm{L}(\mathtt{w}), \mathtt{w} \rangle \curvearrowright \mathtt{w}, \mathrm{R}(\mathtt{w}), \mathtt{0}$.

Proof. By induction on $\mathtt{w}$. We show some representative cases.

Case $\mathtt{w} = \mathtt{keep}(\mathtt{r})$

$\mathrm{L}(\mathtt{keep}(\mathtt{r})) = \mathtt{r}$. Immediate from rule **r-nochange**.

Case $\mathtt{w} = \mathtt{new}(\mathtt{v}, \mathtt{v}')$

$\mathrm{L}(\mathtt{new}(v, v')) = \mathtt{v}$ and $\mathrm{R}(\mathtt{new}(v, v')) = \mathtt{v}'$. Immediate from rule **r-new**.

Case $\mathtt{w} = (\mathtt{w}_1, \mathtt{w}_2)$

By IH on $\mathtt{w}_1$, we get $\langle \mathrm{L}(\mathtt{w}_1), \mathtt{w}_1 \rangle \curvearrowright \mathtt{w}_1, \mathrm{R}(\mathtt{w}_1), \mathtt{0}$ $(\star)$
By IH on $\mathtt{w}_2$, we get $\langle \mathrm{L}(\mathtt{w}_2), \mathtt{w}_2 \rangle \curvearrowright \mathtt{w}_2, \mathrm{R}(\mathtt{w}_2), \mathtt{0}$ $(\dagger)$
Therefore, by instantiating **r-pair** rule using $(\star)$ and $(\dagger)$, we get
$\langle \mathrm{L}((\mathtt{w}_1, \mathtt{w}_2)), (\mathtt{w}_1, \mathtt{w}_2) \rangle \curvearrowright (\mathtt{w}_1, \mathtt{w}_2), \mathrm{R}((\mathtt{w}_1, \mathtt{w}_2)), 0$.

Case $\mathtt{w} = \mathtt{nil}$
Follows immediately from the **r-nochange** rule $\langle \mathtt{nil}, \mathtt{nil} \rangle \curvearrowright \mathtt{nil}, \mathtt{nil}, 0$.

Case $\mathtt{w} = \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)$
By IH on $\mathtt{w}_1$, we get $\langle \mathrm{L}(\mathtt{w}_1), \mathtt{w}_1 \rangle \curvearrowright \mathtt{w}_1, \mathrm{R}(\mathtt{w}_1), \mathtt{0}$ ($\star$)
By IH on $\mathtt{w}_2$, we get $\langle \mathrm{L}(\mathtt{w}_2), \mathtt{w}_2 \rangle \curvearrowright \mathtt{w}_2, \mathrm{R}(\mathtt{w}_2), \mathtt{0}$ (†)
Therefore, by instantiating **r-cons1** rule using ($\star$) and (†), we get
$\langle \mathrm{L}(\mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)), \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2) \rangle \curvearrowright \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2), \mathrm{R}(\mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)), \mathtt{0}$.

Case $\mathtt{w} = \mathtt{fix\ f(x).ee}$
Immediate from rule **r-fix**.

□

Lemma 5 (Value interpretation containment)
The following hold.

1. $(m, \mathtt{w}) \in [\![\tau]\!]_{\mathtt{v}}$ then $(m, \mathtt{w}) \in [\![\tau]\!]^0_\varepsilon$.
2. $(k, v) \in (\!|\tau|\!)_v$ then $(k, v) \in (\!|\tau|\!)^0_\varepsilon$.

Proof of (1). Following the definition of $[\![\tau]\!]^0_\varepsilon$, assume that $\mathrm{L}(\mathtt{w}) \Downarrow \langle \mathtt{v}, \mathtt{D} \rangle, \mathtt{f}$ and $f < m$.
We have to show that there exist $v', \mathtt{w}', \mathtt{D}', \mathtt{c}', \mathtt{f}'$ such that:

1. $\langle \langle v, D \rangle, \mathtt{w} \rangle \curvearrowright \mathtt{w}', \langle \mathtt{v}', \mathtt{D}' \rangle, \mathtt{c}'$
2. $\mathrm{R}(\mathtt{w}') \Downarrow \langle \mathtt{v}', \mathtt{D}' \rangle, \mathtt{f}'$
3. $v = \mathrm{L}(\mathtt{w}') \ \wedge \ \mathtt{v}' = \mathrm{R}(\mathtt{w}')$
4. $c' = 0$
5. $(m - f, \mathtt{w}') \in [\![\tau]\!]_{\mathtt{v}}$

Since $\mathtt{w}$ is a bi-value, $\mathrm{L}(\mathtt{w})$ and $\mathrm{R}(\mathtt{w})$ are values and, hence, $\mathrm{L}(\mathtt{w}) \Downarrow \langle \mathrm{L}(\mathtt{w}), \mathrm{L}(\mathtt{w}) \rangle, \mathtt{0}$ and $\mathrm{R}(\mathtt{w}) \Downarrow \langle \mathrm{R}(\mathtt{w}), \mathrm{R}(\mathtt{w}) \rangle, \mathtt{0}$ (**value** evaluation rule). This forces $v = \mathrm{L}(\mathtt{w})$, $v' = \mathrm{R}(\mathtt{w})$, $D = \mathrm{L}(\mathtt{w})$, $D' = \mathrm{R}(\mathtt{w})$ and $j = 0$. We choose $\mathtt{w}' = \mathtt{w}$. This trivially yields (2), (3) and (5). Next, from Lemma 4, $\langle \mathrm{L}(\mathtt{w}), \mathtt{w} \rangle \curvearrowright \mathtt{w}, \mathrm{R}(\mathtt{w}), \mathtt{0}$. This yields (1) and (4).

□

Proof of (2). Following the definition of $(\!|\tau|\!)^0_\varepsilon$, assume that $0 < k$. Then, we can immediately show

1. $v \Downarrow \langle v, v \rangle, 0$ obtained by **value** evaluation rule.
2. $f = 0 \leq 0$
3. $(k - 0, v) \in (\!|\tau|\!)_v$ which follows from the assumption.

□

Lemma 6 (No input change)
If $\mathrm{L}(\mathtt{ee}) \Downarrow \mathtt{T}, \mathtt{f}$ and $\langle T, \mathtt{ee} \rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$ and $\mathtt{stable}(\mathtt{ee})$ then $\mathtt{stable}(\mathtt{w}')$ and $c' = 0$.

Proof. Immediate from **no-change** replay rule. □

Lemma 7 (Stable context soundness)
Suppose $(\forall x \in \Gamma \quad \Delta; \Phi \models \Gamma(x) \sqsubseteq \square(\Gamma(x)))$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$. Then, the following hold.

1. If $\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa$, then $\texttt{stable}(\theta\ulcorner e \urcorner)$.
2. If $\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{w} \gg \tau$ and $\texttt{stable}(\mathtt{w})$, then $\texttt{stable}(\theta\mathtt{w})$.
3. If $\Delta; \Phi; \Gamma \vdash^\kappa_\epsilon \mathtt{ee} \gg \tau$ and $\texttt{stable}(\mathtt{ee})$, then $\texttt{stable}(\theta\mathtt{ee})$.

Proof. All three statements have similar proofs. We show the proof of (1). By definition, $\ulcorner e \urcorner$ does not have any occurrence of `new`. Therefore, it suffices to show that for any $x \in \Gamma$, $\texttt{stable}(\theta(x))$. Pick any $x \in \Gamma$. From the definition of $\mathcal{G}[\![\sigma\Gamma]\!]$, $(m, \theta(x)) \in [\![\sigma(\Gamma(x))]\!]_v$. By Lemma 11, $(m, \theta(x)) \in [\![\square(\sigma(\Gamma(x)))]\!]_v$. From the definition of $[\![\square(\tau)]\!]_v$, we get $\texttt{stable}(\theta(x))$, as needed. □

Lemma 8 (Stable Type Lemma)
The following hold.

1. If and only if $(m, \mathtt{w}) \in [\![\tau]\!]_\mathtt{v}$ and $\texttt{stable}(\mathtt{w})$, then $(m, \mathtt{w}) \in [\![\square(\tau)]\!]_\mathtt{v}$.
2. If and only if $(m, \mathtt{ee}) \in [\![\tau]\!]^\kappa_\varepsilon$ and $\texttt{stable}(\mathtt{ee})$, then $(m, \mathtt{ee}) \in [\![\square(\tau)]\!]^\kappa_\varepsilon$.

Proof. (1) follows immediately by definition. (2) and (3) are proved by using statement (1).

Proof of statement (2), direction ($\rightarrow$):

Assume that $(m, \mathtt{ee}) \in [\![\tau]\!]^\kappa_\varepsilon$ ($\star$) and $\texttt{stable}(\mathtt{ee})$ ($\star\star$).

TS: $(m, \mathtt{ee}) \in [\![\square(\tau)]\!]^\kappa_\varepsilon$

Assume $f < m$ such that $\mathrm{L}(\mathtt{ee}) \Downarrow \langle \mathtt{v}, \mathtt{D} \rangle, \mathtt{f}$ ($\dagger$) . STS:

1. $\langle T, \mathtt{ee} \rangle \curvearrowright \mathtt{w}', \langle \mathtt{v}', \mathtt{D}' \rangle, \mathtt{c}'$
2. $\mathrm{R}(\mathtt{ee}) \Downarrow \langle \mathtt{v}', \mathtt{D}' \rangle, \mathtt{f}'$
3. $v' = \mathrm{R}(\mathtt{w}') \ \wedge \ \mathtt{v} = \mathrm{L}(\mathtt{w}')$
4. $c' \leq \kappa$
5. $(m - f, \mathtt{w}') \in [\![\square(\tau)]\!]_\mathtt{v}$

By unrolling the ($\star$) with ($\dagger$), we immediately obtain statements 1-4, and $(m - f, \mathtt{w}') \in [\![\tau]\!]_\mathtt{v}$ ($\diamond$). For statement 5, by instantiating IH 1 with direction ($\rightarrow$), STS: $(m - f, \mathtt{w}') \in [\![\tau]\!]_\mathtt{v}$ and $\texttt{stable}(\mathtt{w})$. We conclude the former by ($\diamond$). For the latter, we instantiate Lemma 6 with ($\star\star$), ($\dagger$) and statement 1. to obtain $\texttt{stable}(\mathtt{w})$.

Proof of statement (2), direction ($\leftarrow$):

Assume that $(m, \mathtt{ee}) \in [\![\square(\tau)]\!]^\kappa_\varepsilon$ ($\star$).

TS: $(m, \mathtt{ee}) \in [\![\tau]\!]^\kappa_\varepsilon$ and $\texttt{stable}(\mathtt{ee})$ ($\diamond$)

Assume $f < m$ such that $\mathrm{L}(\mathtt{ee}) \Downarrow \langle \mathtt{v}, \mathtt{D} \rangle, \mathtt{f}$ ($\dagger$). STS:

1. $\langle T, \mathtt{ee} \rangle \curvearrowright \mathtt{w}', \langle \mathtt{v}', \mathtt{D}' \rangle, \mathtt{c}'$
2. $\mathrm{R}(\mathtt{ee}) \Downarrow \langle \mathtt{v}', \mathtt{D}' \rangle, \mathtt{f}'$
3. $v' = \mathrm{R}(\mathtt{w}') \ \wedge \ \mathtt{v} = \mathrm{L}(\mathtt{w}')$

4. $c' \leq \kappa$

5. $(m - f, \mathtt{w}') \in [\![\tau]\!]_{\mathtt{v}}$

By unrolling the ($\star$) with ($\dagger$) , we immediately obtain statements 1-4, and $(m - f, \mathtt{w}') \in [\![\Box(\tau)]\!]_{\mathtt{v}}$ ($\diamond$).
For statement 5. and ($\diamond$), we conclude instantiating IH 1 with direction ($\leftarrow$) on ($\diamond$).

□

Lemma 9 (List variation invariance lemma)
The following hold. If $(m, \mathtt{w}) \in [\![\mathtt{list}\,[\mathtt{n}]^{\alpha}\,\tau]\!]_{\mathtt{v}}$, then $\alpha \leq n$.

Proof. The statement is proved by induction on the bi-value $\mathtt{w}$. There are two cases.

- $(m, \mathtt{nil}) \in [\![\mathtt{list}\,[0]^{0}\,\tau]\!]_{v}$
  This case is immediately true since $0 \leq 0$.
- $(m, \mathtt{cons}(\mathtt{w_1}, \mathtt{w_2})) \in [\![\mathtt{list}\,[\mathtt{n}+1]^{\alpha}\,\tau]\!]_{\mathtt{v}}$
  TS: $\alpha \leq n + 1$.
  There are two cases.
  - $(m, \mathtt{w}) \in [\![\Box(\tau)]\!]_{\mathtt{v}}$ and $(m, \mathtt{w_2}) \in [\![\mathtt{list}\,[\mathtt{n}]^{\alpha}\,\tau]\!]_{\mathtt{v}}$
    By IH on $\mathtt{w_2}$, we get $\alpha \leq n$. Hence, $\alpha \leq n + 1$.
  - $(m, \mathtt{w}) \in [\![\tau]\!]_{\mathtt{v}}$ and $(m, \mathtt{w_2}) \in [\![\mathtt{list}\,[\mathtt{n}]^{\alpha-1}\,\tau]\!]_{\mathtt{v}}$ and $\alpha > 0$
    By IH on $\mathtt{w_2}$, we get $\alpha - 1 \leq n$. Hence, $\alpha \leq n + 1$.

□

Lemma 10 (No-change list variation invariance lemma)
The following hold. If $(m, \mathtt{cons}(\mathtt{w_1}, \mathtt{w_2})) \in [\![\mathtt{list}\,[\mathtt{I}+1]^{\alpha}\,\tau]\!]_{\mathtt{v}}$ and $(m, \mathtt{w_1}) \in [\![\Box(\tau)]\!]_{\mathtt{v}}$, then $\alpha \leq I$.

Proof. The statement is proved by case analysis on the first premise. There are two cases.

- $(m, \mathtt{w}) \in [\![\Box(\tau)]\!]_{\mathtt{v}}$ and $(m, \mathtt{w_2}) \in [\![\mathtt{list}\,[\mathtt{I}]^{\alpha}\,\tau]\!]_{\mathtt{v}}$
  By Lemma 9 on $\mathtt{w_2}$, we immediately get $\alpha \leq I$.
- $(m, \mathtt{w}) \in [\![\tau]\!]_{\mathtt{v}}$ and $(m, \mathtt{w_2}) \in [\![\mathtt{list}\,[\mathtt{n}]^{\alpha-1}\,\tau]\!]_{\mathtt{v}}$ and $\alpha > 0$
  This case contradicts our assumption that $(m, \mathtt{w_1}) \in [\![\Box(\tau)]\!]_{\mathtt{v}}$.

□

Lemma 11 (Bi-value subtyping soundness)
The following hold.

1. If $\Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \mathtt{w}) \in [\![\sigma\mathtt{A}]\!]_{\mathtt{v}}$, then $(m, \mathtt{w}) \in [\![\sigma\mathtt{A}']\!]_{\mathtt{v}}$.
2. If $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \mathtt{w}) \in [\![\sigma\tau]\!]_{\mathtt{v}}$, then $(m, \mathtt{w}) \in [\![\sigma\tau']\!]_{\mathtt{v}}$.
3. If $\Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, v) \in (\!|\sigma A|\!)_{v}$, then $(m, v) \in (\!|\sigma A'|\!)_{v}$.
4. If $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, v) \in (\!|\sigma\tau|\!)_{v}$, then $(m, v) \in (\!|\sigma\tau'|\!)_{v}$.
5. If $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \mathtt{ee}) \in [\![\sigma\tau]\!]^{\kappa}_{\varepsilon}$ and $\kappa \leq \kappa'$, then $(m, \mathtt{ee}) \in [\![\sigma\tau']\!]^{\kappa'}_{\varepsilon}$.

6. If $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, e) \in \ (\!|\sigma\tau|\!)^{\kappa}_{\varepsilon}$ and $\kappa \leq \kappa'$, then $(m, e) \in \ (\!|\sigma\tau'|\!)^{\kappa'}_{\varepsilon}$.

Proof. Proof of statements 1-4 are by induction on the subtyping derivation. First, we show the proof of statements 5-6.

Proof of statement (5):

Assume that $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \texttt{ee}) \in \ [\![\sigma\tau]\!]^{\kappa}_{\varepsilon}$ ($\star$) and $\kappa \leq \kappa'$.

TS: $(m, \texttt{ee}) \in \ [\![\sigma\tau']\!]^{\kappa'}_{\varepsilon}$

By unrolling the definition of $[\![\cdot]\!]^{\cdot}_{\varepsilon}$, assume that $\mathrm{L}(\texttt{ee}) \Downarrow \langle \texttt{v}, \texttt{D} \rangle, \texttt{f}$ (†) and $f < m$ (††).

STS:

1. $\langle \langle v, D \rangle, \texttt{ee} \rangle \curvearrowright \texttt{w}', \langle \texttt{v}', \texttt{D}' \rangle, \texttt{c}'$
2. $\mathrm{R}(\texttt{ee}) \Downarrow \langle \texttt{v}', \texttt{D}' \rangle, \texttt{f}'$
3. $v' = \mathrm{R}(\texttt{w}') \ \wedge \ \texttt{v} = \mathrm{L}(\texttt{w}')$
4. $c' \leq \kappa'$
5. $(m - f, \texttt{w}') \in [\![\tau']\!]_{\texttt{v}}$

By unrolling ($\star$) with (†) and (††), we have

a) $\langle \langle v, D \rangle, \texttt{ee} \rangle \curvearrowright \texttt{w}', \langle \texttt{v}', \texttt{D}' \rangle, \texttt{c}'$

b) $\mathrm{R}(\texttt{ee}) \Downarrow \langle \texttt{v}', \texttt{D}' \rangle, \texttt{f}'$

c) $v' = \mathrm{R}(\texttt{w}') \ \wedge \ \texttt{v} = \mathrm{L}(\texttt{w}')$

d) $c' \leq \kappa$

e) $(m - f, \texttt{w}') \in [\![\tau]\!]_{\texttt{v}}$

Statements 1-3 immediately follow from a)-c).

Statement 4 follows as $c' \leq \kappa'$ since $\kappa \leq \kappa'$ and d).

Statement 5 follows by instantiating IH 2 with e) and the assumption $\Delta; \Phi \models \tau \sqsubseteq \tau'$.

Proof of statement (6):

Assume that $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, e) \in \ (\!|\sigma\tau|\!)^{\kappa}_{\varepsilon}$ ($\star$) and $\kappa \leq \kappa'$.

TS: $(m, e) \in \ (\!|\sigma\tau'|\!)^{\kappa'}_{\varepsilon}$

By unrolling the definition of $(\!|\cdot|\!)^{\cdot}_{\varepsilon}$, assume that $\kappa' < m$

STS:

1. $e \Downarrow \langle v, D \rangle, f$
2. $f \leq \kappa'$
3. $(m - f, v) \in \ (\!|\sigma\tau'|\!)_{v}$

Since we know that $\kappa \leq \kappa' < m$, we also know $\kappa < m$, so we unroll the definition of ($\star$) to obtain

a) $e \Downarrow \langle v, D \rangle, f$

b) $f \leq \kappa$ and

c) $(m-f, v) \in (\!|\sigma\tau|\!)_v$

a) concludes 1. Using b) and $\kappa \leq \kappa'$ we get 2. By IH3 on v using $\Delta;\Phi \models \tau \sqsubseteq \tau'$ and c), we obtain 3.

Proof of statement (1):

Case $\dfrac{\Delta;\Phi \models \tau_1' \sqsubseteq \tau_1 \qquad \Delta;\Phi \models \tau_2 \sqsubseteq \tau_2' \qquad \Delta;\Phi \models \kappa \leq \kappa'}{\Delta;\Phi \models^{\mathsf{A}} \tau_1 \xrightarrow{\delta(\kappa)} \tau_2 \sqsubseteq \tau_1' \xrightarrow{\delta(\kappa')} \tau_2'} \to 2$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \texttt{fix}\ f(x).\texttt{ee}\delta) \in [\![\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2]\!]_{\mathtt{v}}$

There are two cases.

subcase 1: $\delta = \mathbb{S}$

We have $(m, \texttt{fix}\ f(x).\texttt{ee}) \in [\![\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2]\!]_{\mathtt{v}}$ ($\star$)

TS: $(m, \texttt{fix}\ f(x).\texttt{ee}) \in [\![\sigma\tau_1' \xrightarrow{\mathbb{S}(\sigma\kappa')} \sigma\tau_2']\!]_{\mathtt{v}}$

Let $F = \texttt{fix}\ f(x).\texttt{ee}$.

Pick $j < m$ s.t. $(j, \mathtt{w}) \in [\![\sigma\tau_1']\!]_{\mathtt{v}}$ ($\dagger$). STS: $(j, \texttt{ee}[\mathtt{F/f}, \mathtt{w/x}]) \in [\![\sigma\tau_2']\!]_{\varepsilon}^{\sigma\kappa'}$.

By IH 2 on ($\dagger$) and the premise $\Delta;\Phi \models \tau_1' \sqsubseteq \tau_1$, we get $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$.

By unrolling ($\star$), we get $(j, \texttt{ee}[\mathtt{F/f}, \mathtt{w/x}]) \in [\![\sigma\tau_2]\!]_{\varepsilon}^{\sigma\kappa}$($\dagger\dagger$).

We conclude by IH 4 on ($\dagger\dagger$) using $\Delta;\Phi \models \tau_2 \sqsubseteq \tau_2'$ and $\Delta;\Phi \models \kappa \leq \kappa'$.

subcase 2: $\delta = \mathbb{C}$

We have $(m, \texttt{fix}\ f(x).\texttt{ee}) \in [\![\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2]\!]_{\mathtt{v}}$ ($\star\star$)

TS: $(m, \texttt{fix}\ f(x).\texttt{ee}) \in [\![\sigma\tau_1' \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2']\!]_{\mathtt{v}}$

The are two subcases:

subsubcase 1: TS: $\forall k.\ (k, \texttt{fix}\ f(x).\mathrm{R}(\texttt{ee})) \in (\!|\sigma\tau_1' \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2'|\!)_{\mathtt{v}}$

Assume k and let $F' = \texttt{fix}\ f(x).\mathrm{R}(\texttt{ee})$.

Pick $j < m$ s.t. $(j, v) \in (\!|\sigma\tau_1'|\!)_v$ ($\dagger$). STS: $(j, \mathrm{R}(\texttt{ee})[\mathtt{F'/f}, \mathtt{v/x}]) \in (\!|\sigma\tau_2'|\!)_{\varepsilon}^{\sigma\kappa'}$.

By IH 4 on ($\dagger$) and the premise $\Delta;\Phi \models \tau_1' \sqsubseteq \tau_1$, we get $(j, v) \in (\!|\sigma\tau_1|\!)_v$.

By unrolling first part of ($\star\star$), we get $(j, \mathrm{R}(\texttt{ee})[\mathtt{F'/f}, \mathtt{v/x}]) \in (\!|\sigma\tau_2|\!)_{\varepsilon}^{\sigma\kappa}$ ($\dagger\dagger$).

We conclude by IH 5 on ($\dagger\dagger$) using $\Delta;\Phi \models \tau_2 \sqsubseteq \tau_2'$ and $\Delta;\Phi \models \kappa \leq \kappa'$.

subsubcase 2: Pick $j < m$ s.t. $(j, \mathtt{w}) \in [\![\sigma\tau_1']\!]_{\mathtt{v}}$ ($\dagger$). STS: $(j, \texttt{ee}[\mathtt{F/f}, \mathtt{w/x}]) \in [\![\sigma\tau_2']\!]_{\varepsilon}^{\sigma\kappa'}$.

By IH 2 on ($\dagger$) and the premise $\Delta;\Phi \models \tau_1' \sqsubseteq \tau_1$, we get $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$.

By unrolling second part of ($\star\star$), we get $(j, \texttt{ee}[\mathtt{F/f}, \mathtt{w/x}]) \in [\![\sigma\tau_2]\!]_{\varepsilon}^{\sigma\kappa}$($\dagger\dagger$).

We conclude by IH 5 on ($\dagger\dagger$) using $\Delta;\Phi \models \tau_2 \sqsubseteq \tau_2'$ and $\Delta;\Phi \models \kappa \leq \kappa'$.

Case $\dfrac{t :: S, \Delta;\Phi \models \tau \sqsubseteq \tau' \qquad t :: S, \Delta;\Phi \models \kappa \leq \kappa'}{\Delta;\Phi \models^{\mathsf{A}} \forall t \overset{\delta(\kappa)}{::} S.\ \tau \sqsubseteq \forall t \overset{\delta(\kappa')}{::} S.\ \tau'} \forall 1$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \mathtt{w}) \in [\![\forall \mathtt{t} \overset{\delta(\sigma\kappa)}{::} \mathtt{S}.\ \sigma\tau]\!]_{\mathtt{v}}$ and $\models \sigma\Phi$.

TS: $(m, \mathtt{w}) \in [\![\forall \mathtt{t} \overset{\delta(\sigma\kappa')}{::} \mathtt{S}.\ \sigma\tau']\!]_{\mathtt{v}}$.

We case analyze δ.

subcase 1: $\delta = \mathbb{S}$

TS: $(m, \mathtt{w}) \in [\![\forall \mathtt{t} \overset{\mathbb{S}(\sigma\kappa')}{::} \mathtt{S}.\ \sigma\tau']\!]_{\mathtt{v}}$

Assume that $\vdash I :: S$ ($\star$).

STS: $(m, \mathtt{ee}) \in [\![\sigma\tau'\{\mathtt{I}/\mathtt{t}\}]\!]_{\varepsilon}^{\sigma\kappa'\{\mathtt{I}/\mathtt{t}\}}$ ($\dagger$).

We have $(m, \Lambda.\mathtt{ee}) \in [\![\forall \mathtt{t} \overset{\mathbb{S}(\sigma\kappa)}{::} \mathtt{S}.\ \sigma\tau]\!]_{\mathtt{v}}$.

Unrolling the definition of $[\![\cdot]\!]_v$ with the assumption ($\star$), we get $(m, \mathtt{ee}) \in [\![\sigma\tau\{\mathtt{I}/\mathtt{t}\}]\!]_{\varepsilon}^{\sigma\kappa\{\mathtt{I}/\mathtt{t}\}}$ ($\star\star$).

By instantiating IH 6 with $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$ and ($\star\star$), we get:

$(m, \mathtt{ee}) \in [\![\sigma[\mathtt{t} \mapsto \mathtt{I}]\tau']\!]_{\varepsilon}^{\sigma[\mathtt{t}\mapsto\mathtt{I}]\kappa'}$ which is same as ($\dagger$).

subcase 2: $\delta = \mathbb{C}$

$(m, \mathtt{w}) \in [\![\forall \mathtt{t} \overset{\mathbb{C}(\sigma\kappa)}{::} \mathtt{S}.\ \sigma\tau]\!]_{\mathtt{v}}$ ($\diamond$)

TS: $(m, \mathtt{w}) \in [\![\forall \mathtt{t} \overset{\mathbb{C}(\sigma\kappa')}{::} \mathtt{S}.\ \sigma\tau']\!]_{\mathtt{v}}$

Assume that $\vdash I :: S$ ($\star$).

subsubcase 1: STS: $\forall k.\ (k, \mathrm{R}(\mathtt{ee})) \in (\!|\sigma\tau'\{\mathtt{I}/\mathtt{t}\}|\!)_{\varepsilon}^{\sigma\kappa'\{\mathtt{I}/\mathtt{t}\}}$ ($\dagger$).

Pick k. Then STS: $(k, \mathrm{R}(\mathtt{ee})) \in (\!|\sigma\tau'\{\mathtt{I}/\mathtt{t}\}|\!)_{\varepsilon}^{\sigma\kappa'\{\mathtt{I}/\mathtt{t}\}}$ ($\dagger\dagger$).

By unrolling first part of ($\diamond$) with k and ($\star$), we get:

$(k, \mathrm{R}(\mathtt{ee})) \in (\!|\sigma\tau\{\mathtt{I}/\mathtt{t}\}|\!)_{\varepsilon}^{\sigma\kappa\{\mathtt{I}/\mathtt{t}\}}$ ($\star\star$).

By instantiating IH 5 with $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$ and ($\star\star$), we get:

$(k, \mathrm{R}(\mathtt{ee})) \in (\!|\sigma[\mathtt{t} \mapsto \mathtt{I}]\tau'|\!)_{\varepsilon}^{\sigma[\mathtt{t}\mapsto\mathtt{I}]\kappa'}$ which is same as ($\dagger$).

subsubcase 2: STS: $(m, \mathtt{ee}) \in [\![\sigma\tau'\{\mathtt{I}/\mathtt{t}\}]\!]_{\varepsilon}^{\sigma\kappa'\{\mathtt{I}/\mathtt{t}\}}$ ($\dagger$).

We have $(m, \Lambda.\mathtt{ee}) \in [\![\forall \mathtt{t} \overset{\mathbb{S}(\sigma\kappa)}{::} \mathtt{S}.\ \sigma\tau]\!]_{\mathtt{v}}$.

By unrolling second part of ($\diamond$) with the assumption ($\star$), we get $(m, \mathtt{ee}) \in [\![\sigma\tau\{\mathtt{I}/\mathtt{t}\}]\!]_{\varepsilon}^{\sigma\kappa\{\mathtt{I}/\mathtt{t}\}}$ ($\star\star$).

By instantiating IH 5 with $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$ and ($\star\star$), we get:

$(m, \mathtt{ee}) \in [\![\sigma[\mathtt{t} \mapsto \mathtt{I}]\tau']\!]_{\varepsilon}^{\sigma[\mathtt{t}\mapsto\mathtt{I}]\kappa'}$ which is same as ($\dagger$).

Case $\dfrac{\Delta; \Phi \models \alpha \doteq 0}{\Delta; \Phi \models_{\mathbb{S}} \mathtt{list}\,[n]^{\alpha}\ \tau \equiv \mathtt{list}\,[n]^{\alpha}\ \Box(\tau)}$ **l1***

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \mathtt{w}) \in [\![\mathtt{list}\,[\mathtt{n}]^{\alpha}\ \tau]\!]_{\mathtt{v}}$.

TS: $(m, \mathtt{w}) \in [\![\mathtt{list}\,[\sigma\mathtt{n}]^{\sigma\alpha}\ \Box(\sigma\tau)]\!]_{\mathtt{v}}$

We prove the following more general statement by subinduction on $\mathtt{w}'$:

For all $\mathtt{w}'$, if $(m, \mathtt{w}') \in [\![\mathtt{list}\,[\sigma\mathtt{n}]^{\sigma\alpha}\ \sigma\tau]\!]_{\mathtt{v}}$ ($\star$) then $(m, \mathtt{w}') \in [\![\mathtt{list}\,[\sigma\mathtt{n}]^{\sigma\alpha}\ \Box(\sigma\tau)]\!]_{\mathtt{v}}$. By the premise, $\sigma\alpha = 0$.

subcase 1: $\mathtt{w}' = \mathtt{nil}$

From the assumption marked ($\star$), we get $(m, \mathtt{nil}) \in [\![\mathtt{list}\,[\sigma n]^{0}\ \sigma\tau]\!]_{v}$.

Therefore, we have $\sigma n = 0$, and $(m, \mathtt{nil}) \in [\![\mathtt{list}\,[\sigma n]^0\ \Box(\sigma\tau)]\!]_v$ by definition.

subcase 2: $\mathtt{w}' = \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)$

From the assumption marked ($\star$), we get $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\sigma \mathtt{n}]^0\ \sigma\tau]\!]_{\mathtt{v}}$.

Therefore, $\exists I.\ \sigma n = I + 1$.

We have $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{I}+\mathtt{1}]^0\ \sigma\tau]\!]_{\mathtt{v}}$.

TS: $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{I}+\mathtt{1}]^0\ \Box(\sigma\tau)]\!]_{\mathtt{v}}$.

We have two possible cases:

- $(m, \mathtt{w}_1) \in [\![\Box(\sigma\tau)]\!]_{\mathtt{v}}$ (†) and $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{I}]^0\ \sigma\tau]\!]_{\mathtt{v}}$ (††).
  By sub-IH on (††), we get $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{I}]^0\ \Box(\sigma\tau)]\!]_{\mathtt{v}}$.
  Combining the (†) with the previous statement, we get $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{I}+\mathtt{1}]^0\ \Box(\sigma\tau)]\!]_{\mathtt{v}}$.
- $(m, \mathtt{w}_1) \in [\![\sigma\tau]\!]_{\mathtt{v}}$ ($\diamond$) and $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{I}]^{0-1}\ \sigma\tau]\!]_{\mathtt{v}}$ ($\diamond\diamond$).
  This case is impossible since $0 - 1 \not\geq 0$.

Case $\dfrac{\Delta; \Phi \models n \doteq n' \qquad \Delta; \Phi \models \alpha \leq \alpha' \leq n \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi \models \mathtt{list}\,[n]^{\alpha}\ \tau \sqsubseteq \mathtt{list}\,[n']^{\alpha'}\ \tau'}$ **l2**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \mathtt{w}) \in [\![\mathtt{list}\,[\sigma \mathtt{n}]^{\sigma\alpha}\ \sigma\tau]\!]_{\mathtt{v}}$ and $\models \sigma\Phi$.

TS: $(m, \mathtt{w}) \in [\![\mathtt{list}\,[\sigma \mathtt{n}']^{\sigma\alpha'}\ \sigma\tau']\!]_{\mathtt{v}}$.

From Assumption 12 applied to the first premise, $\sigma n = \sigma n'$. Therefore,

STS: $(m, \mathtt{w}) \in [\![\mathtt{list}\,[\sigma \mathtt{n}]^{\sigma\alpha'}\ \sigma\tau']\!]_{\mathtt{v}}$.

From Assumption 12 applied to the second premise, we get: $\sigma\alpha \leq \sigma\alpha'$. Therefore, it suffices to prove the following more general statement.

For all $\mathtt{w}', \mathtt{l}, \beta, \gamma$, if $\beta \leq \gamma \leq l$ and $(m, \mathtt{w}') \in [\![\mathtt{list}\,[\mathtt{l}]^{\beta}\ \sigma\tau]\!]_{\mathtt{v}}$, then $(m, \mathtt{w}') \in [\![\mathtt{list}\,[\mathtt{l}]^{\gamma}\ \sigma\tau']\!]_{\mathtt{v}}$.

We establish this statement by subinduction on $\mathtt{w}'$.

subcase 1: $\mathtt{w}' = \mathtt{nil}$

The assumption $(m, \mathtt{nil}) \in [\![\mathtt{list}\,[l]^{\beta}\ \sigma\tau]\!]_v$ forces $l = \beta = 0$. Since $\beta \leq \gamma \leq l$, $\gamma = l = 0$, $(m, \mathtt{w}') \in [\![\mathtt{list}\,[\mathtt{l}]^{\gamma}\ \sigma\tau']\!]_{\mathtt{v}}$ follows immediately by definition.

subcase 2: $\mathtt{w}' = \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)$

By assumption, $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{l}]^{\beta}\ \sigma\tau]\!]_{\mathtt{v}}$. Therefore, $l = l' + 1$ for some l'.

TS: $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{l}'+\mathtt{1}]^{\gamma}\ \sigma\tau']\!]_{\mathtt{v}}$.

There are two possible cases for $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{l}]^{\beta}\ \sigma\tau]\!]_{\mathtt{v}}$.

- $(m, \mathtt{w}_1) \in [\![\Box(\sigma\tau)]\!]_{\mathtt{v}}$ (†) and $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{l}']^{\beta}\ \sigma\tau]\!]_{\mathtt{v}}$ (††).
  By Lemma 9 using (††), we get $\beta \leq l'$ ($\star$).
  Note that we assumed $\beta \leq \gamma \leq l' + 1$. There are two cases for γ.
  - $\gamma \leq l'$
    Using $\beta \leq \gamma$ (by main assumption) and $\gamma \leq l'$, we obtain $\beta \leq \gamma \leq l'$ and instantiate the sub-IH with $\mathtt{w}_2$ to get $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{l}']^{\gamma}\ \sigma\tau']\!]_{\mathtt{v}}$.

By IH 2 on $\Delta;\Phi \models \tau \sqsubseteq \tau'$ with (†), we get $(m, \mathtt{w}_1) \in [\![\Box(\sigma\tau')]\!]_{\mathtt{v}}$.
Therefore, we can conclude that $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{l}'+1]^{\gamma}\ \sigma\tau']\!]_{\mathtt{v}}$.

– $\gamma = l' + 1$
Since $\beta \leq l'$ by (⋆) and $l' = \gamma - 1$, we obtain $\beta \leq \gamma - 1 \leq l'$ and instantiate the sub-IH with $\mathtt{w}_2$ to get $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{l}']^{\gamma-1}\ \sigma\tau']\!]_{\mathtt{v}}$.
By IH 2 on $\Delta;\Phi \models \tau \sqsubseteq \tau'$ with (†), we get $(m, \mathtt{w}_1) \in [\![\Box(\sigma\tau')]\!]_{\mathtt{v}}$. By Lemma 8, we get $\in [\![\sigma\tau']\!]_v$.
Therefore, we can conclude that $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{l}'+1]^{\gamma}\ \sigma\tau']\!]_{\mathtt{v}}$.

By the sub-IH on (††), we get $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{l}']^{\beta'}\ \sigma\tau']\!]_{\mathtt{v}}$.
Combining, we get $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{l}'+1]^{\beta'}\ \sigma\tau']\!]_{\mathtt{v}}$.

- $(m, \mathtt{w}_1) \in [\![\sigma\tau]\!]_{\mathtt{v}}$ (⋄) and $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{l}']^{\beta-1}\ \sigma\tau]\!]_{\mathtt{v}}$ (⋄⋄).
By IH on $\Psi;\Delta;\Phi \models \tau \sqsubseteq \tau'$ with (⋄), we get $(m, \mathtt{w}_1) \in [\![\sigma\tau']\!]_{\mathtt{v}}$
By the sub-IH on (⋄⋄) using $\beta - 1 \leq \gamma - 1 \leq l'$, we get $(m, \mathtt{w}_2) \in [\![\mathtt{list}\,[\mathtt{l}']^{\gamma-1}\ \sigma\tau']\!]_{\mathtt{v}}$.
Combining these two yields $(m, \mathtt{cons}(\mathtt{w}_1, \mathtt{w}_2)) \in [\![\mathtt{list}\,[\mathtt{l}'+1]^{\gamma}\ \sigma\tau']\!]_{\mathtt{v}}$.

Proof of statement (2):

Case $\dfrac{\Delta;\Phi \models^{\mathtt{A}} A \sqsubseteq A'}{\Delta;\Phi \models (A)^{\mu} \sqsubseteq (A')^{\mu}}\ \mathbf{C}$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \mathtt{w}) \in [\![(\sigma\mathtt{A})^{\mu}]\!]_{\mathtt{v}}$ and $\models \sigma\Phi$.
TS: $(m, \mathtt{w}) \in [\![(\sigma\mathtt{A}')^{\mu}]\!]_{\mathtt{v}}$
In two cases of μ, we have $(m, \mathtt{w}) \in [\![\sigma\mathtt{A}]\!]_{\mathtt{v}}$, STS: $(m, \mathtt{w}) \in [\![\sigma\mathtt{A}']\!]_{\mathtt{v}}$. This immediately follows by instantiating IH 1.

Case $\dfrac{}{\Delta;\Phi \models_{\mathbb{S}} \Box((\tau_1 + \tau_2)^{\mu}) \equiv (\Box(\tau_1) + \Box(\tau_2))^{\mathbb{S}}}\ +\Box$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \mathtt{w}) \in [\![\Box((\sigma\tau_1 + \sigma\tau_2)^{\mu})]\!]_{\mathtt{v}}$.
TS: $(m, \mathtt{w}) \in [\![(\Box(\sigma\tau_1) + \Box(\sigma\tau_2))^{\mathbb{S}}]\!]_{\mathtt{v}}$
There are two cases:

subcase 1: $\mathtt{w} = \mathtt{inl}\ \mathtt{w}'$ and $(m, \mathtt{w}') \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$ and $\mathtt{stable}(\mathtt{w}')$.
By Lemma 8, we have $(m, \mathtt{w}') \in [\![\Box(\sigma\tau_1)]\!]_{\mathtt{v}}$.
Then, it follows that $(m, \mathtt{inl}\ \mathtt{w}') \in [\![\Box(\sigma\tau_1) + \Box(\sigma\tau_2)]\!]_{\mathtt{v}}$

subcase 2: $\mathtt{w} = \mathtt{inr}\ \mathtt{w}'$ and $(m, \mathtt{w}') \in [\![\sigma\tau_2]\!]_{\mathtt{v}}$ and $\mathtt{stable}(\mathtt{w}')$.
By Lemma 8, we have $(m, \mathtt{w}') \in [\![\Box(\sigma\tau_2)]\!]_{\mathtt{v}}$.
Then, it follows that $(m, \mathtt{inr}\ \mathtt{w}') \in [\![(\Box(\sigma\tau_1) + \Box(\sigma\tau_2))^{\mathbb{S}}]\!]_{\mathtt{v}}$

Case $\dfrac{}{\Delta;\Phi \models \Box((\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mu}) \sqsubseteq (\Box(\tau_1) \xrightarrow{\delta(\kappa)} \Box(\tau_2))^{\mathbb{S}}}\ \to 1$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \mathtt{w}) \in [\![\Box(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)]\!]_{\mathtt{v}}$

TS: $(m, \mathtt{w}) \in [\![(\Box(\sigma\tau_1) \xrightarrow{\delta(\sigma\kappa)} \Box(\sigma\tau_2))^{\mathbb{S}}]\!]_{\mathtt{v}}$ There are two cases:

subcase 1: $\delta = \mathbb{S}$

We have $(m, \mathtt{fix}\ f(x).\mathtt{ee}) \in [\![\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2]\!]_{\mathtt{v}}$ (†) and $\mathtt{stable}(\mathtt{fix}\ f(x).\mathtt{ee})$.

TS: $(m, \mathtt{fix}\ f(x).\mathtt{ee}) \in [\![(\Box(\sigma\tau_1) \xrightarrow{\mathbb{S}(\sigma\kappa)} \Box(\sigma\tau_2))^{\mathbb{S}}]\!]_{\mathtt{v}}$

Let $F = \mathtt{fix}\ f(x).\mathtt{ee}$.

Pick $j < m$ s.t. $(j, \mathtt{w}) \in [\![\Box(\sigma\tau_1)]\!]_{\mathtt{v}}$ (⋆). STS: $(j, \mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}]) \in [\![\Box(\sigma\tau_2)]\!]^{\sigma\kappa}_{\varepsilon}$. By definition of (⋆), we get $\mathtt{stable}(\mathtt{w})$ (⋆⋆) and $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$ (⋆⋆⋆). Then, by unrolling (†) with (⋆⋆⋆) where $j < m$, we get $(j, \mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}]) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$. Next, we immediately instantiate Lemma 8 on $\mathtt{stable}(\mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}])$ since $\mathtt{stable}(F)$ and $\mathtt{stable}(\mathtt{w})$, we conclude that $(j, \mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}]) \in [\![\Box(\sigma\tau_2)]\!]^{\sigma\kappa}_{\varepsilon}$.

subcase 2: $\delta = \mathbb{C}$

We have $(m, \mathtt{fix}\ f(x).\mathtt{ee}) \in [\![\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2]\!]_{\mathtt{v}}$ (†) and $\mathtt{stable}(\mathtt{fix}\ f(x).\mathtt{ee})$(◇).

TS: $(m, \mathtt{fix}\ f(x).\mathtt{ee}) \in [\![(\Box(\sigma\tau_1) \xrightarrow{\mathbb{C}(\sigma\kappa)} \Box(\sigma\tau_2))^{\mathbb{S}}]\!]_{\mathtt{v}}$

There are two cases

subsubcase 1: STS: $\forall k.\ (k, \mathrm{R}(\mathtt{fix}\ f(x).\mathtt{ee})) \in (\!|\Box(\sigma\tau_1) \xrightarrow{\mathbb{C}(\sigma\kappa)} \Box(\sigma\tau_2)|\!)_{\mathtt{v}}$ (††)

Pick k. STS: $(k, \mathtt{fix}\ f(x).\mathrm{R}(\mathtt{ee})) \in (\!|\Box(\sigma\tau_1) \xrightarrow{\mathbb{C}(\sigma\kappa)} \Box(\sigma\tau_2)|\!)_{\mathtt{v}}$

Pick $j < k$ s.t. $(j, v) \in (\!|\Box(\sigma\tau_1)|\!)_v = (\!|\sigma\tau_1|\!)_v$ (⋆).

STS: $(j, \mathrm{R}(\mathtt{ee})[\mathtt{fix}\ \mathtt{f}(\mathtt{x}).\mathrm{R}(\mathtt{ee})/\mathtt{f}, \mathtt{v}/\mathtt{x}]) \in (\!|\Box(\sigma\tau_2)|\!)^{\sigma\kappa}_{\varepsilon}$.

By unrolling (†) with (⋆), we get $(j, \mathrm{R}(\mathtt{ee})[\mathtt{fix}\ \mathtt{f}(\mathtt{x}).\mathrm{R}(\mathtt{ee})/\mathtt{f}, \mathtt{v}/\mathtt{x}]) \in (\!|\sigma\tau_2|\!)^{\sigma\kappa}_{\varepsilon} = (\!|\Box(\sigma\tau_2)|\!)^{\sigma\kappa}_{\varepsilon}$

subsubcase 2: Pick $j < m$ s.t. $(j, \mathtt{w}) \in [\![\Box(\sigma\tau_1)]\!]_{\mathtt{v}}$ (⋆).

STS: $(j, \mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}]) \in [\![\Box(\sigma\tau_2)]\!]^{\sigma\kappa}_{\varepsilon}$.

By definition of (⋆), we get $\mathtt{stable}(\mathtt{w})$ (⋆⋆) and $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$ (⋆⋆⋆). Then, by unrolling (†) with (⋆⋆⋆) where $j < m$, we get $(j, \mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}]) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$. Next, we immediately instantiate Lemma 8 on $\mathtt{stable}(\mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}])$ since $\mathtt{stable}(F)$ and $\mathtt{stable}(\mathtt{w})$, we conclude that $(j, \mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}]) \in [\![\Box(\sigma\tau_2)]\!]^{\sigma\kappa}_{\varepsilon}$.

Case $\dfrac{}{\Delta; \Phi \models \Box(\tau) \sqsubseteq \tau}\ \mathbf{T}$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \mathtt{w}) \in [\![\Box(\sigma\tau)]\!]_{\mathtt{v}}$ and $\models \sigma\Phi$.

TS: $(m, \mathtt{w}) \in [\![\sigma\tau]\!]_{\mathtt{v}}$. This immediately follows from the definition.

Case $\dfrac{\Delta; \Phi \models \mu \leq \mu'}{\Delta; \Phi \models (A)^{\mu} \sqsubseteq (A)^{\mu'}}\ \mu$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \mathtt{w}) \in [\![(\sigma\mathtt{A})^{\mu}]\!]_{\mathtt{v}}$ and $\models \sigma\Phi$.

TS: $(m, \mathtt{w}) \in [\![(\sigma\mathtt{A})^{\mu'}]\!]_{\mathtt{v}}$

There are four cases. Two of these are trivial since $\mu = \mu'$. The remaining two are:

subcase 1: $\mu = \mathbb{S}$ and $\mu' = \mathbb{C}$

We have $(m, \mathtt{w}) \in [\![(\sigma \mathtt{A})^{\mathbb{S}}]\!]_{\mathtt{v}} = [\![\sigma \mathtt{A}]\!]_{\mathtt{v}}$ ($\star$).

TS: $(m, \mathtt{w}) \in [\![(\sigma \mathtt{A})^{\mathbb{C}}]\!]_{\mathtt{v}}$.

Since $[\![\sigma A]\!]_v \subseteq [\![(\sigma A)^{\mathbb{C}}]\!]_v$, it follows immediately by ($\star$).

subcase 2: $\mu = \mathbb{C}$ and $\mu' = \mathbb{S}$

This contradicts the assumption that $\mu \leq \mu'$ since $\mathbb{C} \not\leq \mathbb{S}$

Proof of statement (3):

Case $\dfrac{\Delta; \Phi \models \tau_1' \sqsubseteq \tau_1 \qquad \Delta; \Phi \models \tau_2 \sqsubseteq \tau_2' \qquad \Delta; \Phi \models \kappa \leq \kappa'}{\Delta; \Phi \models^{\mathbf{A}} \tau_1 \xrightarrow{\delta(\kappa)} \tau_2 \sqsubseteq \tau_1' \xrightarrow{\delta(\kappa')} \tau_2'} \to 2$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \mathtt{fix}\ f(x).\, e\delta) \in (\!|\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2|\!)_v$

There are two cases.

subcase 1: $\delta = \mathbb{S}$

This is trivial since any function $\mathtt{fix}\ f(x).\, e \in (\!|\sigma\tau_1' \xrightarrow{\mathbb{S}(\sigma\kappa')} \sigma\tau_2'|\!)_v$.

subcase 2: $\delta = \mathbb{C}$

We have $(m, \mathtt{fix}\ f(x).\, e) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$ ($\star\star$)

TS: $(m, \mathtt{fix}\ f(x).\, e) \in (\!|\sigma\tau_1' \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2'|\!)_v$

Let $F = \mathtt{fix}\ f(x).e$.

Pick $j < m$ s.t. $(j, v) \in (\!|\sigma\tau_1'|\!)_v$ ($\dagger$). STS: $(j, e[F/f, v/x]) \in (\!|\sigma\tau_2'|\!)_\varepsilon^{\sigma\kappa'}$.

By IH 4 on ($\dagger$) and the premise $\Delta; \Phi \models \tau_1' \sqsubseteq \tau_1$, we get $(j, v) \in (\!|\sigma\tau_1|\!)_v$.

By unrolling ($\star\star$), we get $(j, e[F/f, v/x]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma\kappa}$ ($\dagger\dagger$).

We conclude by IH 6 on ($\dagger\dagger$) using $\Delta; \Phi \models \tau_2 \sqsubseteq \tau_2'$ and $\Delta; \Phi \models \kappa \leq \kappa'$.

Case $\dfrac{t :: S, \Delta; \Phi \models \tau \sqsubseteq \tau' \qquad t :: S, \Delta; \Phi \models \kappa \leq \kappa'}{\Delta; \Phi \models^{\mathbf{A}} \forall t \overset{\delta(\kappa)}{::} S.\ \tau \sqsubseteq \forall t \overset{\delta(\kappa')}{::} S.\ \tau'} \forall 1$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \Lambda.\, e) \in (\!|\forall t \overset{\delta(\sigma\kappa)}{::} S.\ \sigma\tau|\!)_v$ and $\models \sigma\Phi$.

TS: $(m, \Lambda.\, e) \in (\!|\forall t \overset{\delta(\sigma\kappa')}{::} S.\ \sigma\tau'|\!)_v$.

We case analyze δ.

subcase 1: $\delta = \mathbb{S}$

This is trivial since any $(m, \Lambda.\, e) \in (\!|\forall t \overset{\mathbb{S}(\sigma\kappa')}{::} S.\ \sigma\tau'|\!)_v$.

subcase 2: $\delta = \mathbb{C}$

We have $(m, \Lambda.\, e) \in (\!|\forall t \overset{\mathbb{C}(\sigma\kappa)}{::} S.\ \sigma\tau|\!)_v$ ($\diamond$)

TS: $(m, \Lambda.\, e) \in (\!|\forall t \overset{\mathbb{C}(\sigma\kappa')}{::} S.\ \sigma\tau'|\!)_v$

Assume that $\vdash I :: S$ ($\star$).

STS:$(m, e) \in (\!|\sigma\tau'\{I/t\}|\!)_\varepsilon^{\sigma\kappa'\{I/t\}}$ ($\dagger$).

By unrolling ($\diamond$) with k and ($\star$), we get:

$(m, e) \in (\!|\sigma\tau\{I/t\}|\!)_\varepsilon^{\sigma\kappa\{I/t\}}$ ($\star\star$).

By instantiating IH 6 with $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$ and ($\star\star$), we get:

$(m, e) \in (\!|\sigma[t \mapsto I]\tau'|\!)_\varepsilon^{\sigma[t\mapsto I]\kappa'}$ which is same as ($\dagger$).

Proof of statement (4):

Case $\dfrac{\Delta; \Phi \models \mu \leq \mu'}{\Delta; \Phi \models (A)^{\mu} \sqsubseteq (A)^{\mu'}}\ \mu$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, v) \in (\!|(\sigma A)^{\mu}|\!)_v = (\!|\sigma A|\!)_v$ ($\star$) and $\models \sigma\Phi$.

TS: $(m, v) \in (\!|(\sigma A)^{\mu'}|\!)_v = (\!|\sigma A|\!)_v$.

Since unary relations don't take the mode into account, we immediately conclude by ($\star$).

Case $\dfrac{\Delta; \Phi \models n \doteq n' \qquad \Delta; \Phi \models \alpha \leq \alpha' \leq n \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi \models \mathtt{list}\,[n]^{\alpha}\ \tau \sqsubseteq \mathtt{list}\,[n']^{\alpha'}\ \tau'}$ **l2**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, v) \in (\!|\mathtt{list}\,[\sigma n]^{\sigma\alpha}\ \tau|\!)_v$ and $\models \sigma\Phi$.

TS: $(m, v) \in (\!|\mathtt{list}\,[\sigma n']^{\sigma\alpha'}\ \tau'|\!)_v$.

From Assumption 12 applied to the first premise, $\sigma n = \sigma n'$. Therefore,

STS: $(m, v) \in (\!|\mathtt{list}\,[\sigma n]^{\sigma\alpha'}\ \tau'|\!)_v$.

Note that the unary relation doesn't take the α/ α' into account. Therefore, it suffices to prove the following more general statement.

For all v', l, if $(m, v') \in [\![\mathtt{list}\,[l]^{-}\ \sigma\tau]\!]_v$, then $(m, v') \in [\![\mathtt{list}\,[l]^{-}\ \sigma\tau']\!]_v$.

We establish this statement by subinduction on v'.

subcase 1: $v' = \mathtt{nil}$

The conclusion $(m, \mathtt{nil}) \in [\![\mathtt{list}\,[l]^{-}\ \sigma\tau']\!]_v$ follows immediately by definition.

subcase 2: $v' = \mathtt{cons}(v_1, v_2)$

By assumption, $(m, \mathtt{cons}(v_1, v_2)) \in [\![\mathtt{list}\,[l]^{-}\ \sigma\tau]\!]_v$. Therefore, $l = l' + 1$ for some l' and $(m, v_1) \in (\!|\sigma\tau|\!)_v$ ($\dagger$) and $(m, v_2) \in (\!|\mathtt{list}\,[l']^{-}\ \sigma\tau|\!)_v$ ($\dagger\dagger$) .

TS: $(m, \mathtt{cons}(v_1, v_2)) \in [\![\mathtt{list}\,[l' + 1]^{-}\ \sigma\tau']\!]_v$.

By IH 4 on $\Delta; \Phi \models \tau \sqsubseteq \tau'$ with ($\dagger$), we get $(m, v_1) \in (\!|\sigma\tau'|\!)_v$.

By the sub-IH on ($\dagger\dagger$), we get $(m, v_2) \in (\!|\mathtt{list}\,[l']^{-}\ \sigma\tau'|\!)_v$.

Combining, we get $(m, \mathtt{cons}(v_1, v_2)) \in (\!|\mathtt{list}\,[l' + 1]^{-}\ \sigma\tau'|\!)_v$.

□

We assume that the constraint judgment $\Delta; \Phi \models C$ satisfies some standard properties.

Assumption 12 (Constraint conditions)
The following hold.

1. [Subst1] If $\Delta, i :: S; \Phi \models C$ and $\Delta \vdash I :: S$, then $\Delta; \Phi[I/i] \models C[I/i]$.

2. [Subst2] If $\Delta; \Phi \models C$ and $\Delta; \Phi \wedge C \models C'$, then $\Delta; \Phi \models C'$.

3. [Neg] $\Delta; \Phi \models \neg C$ iff $\Delta; \Phi \not\models C$.

4. [Corr1] If $\models n_1 \leq n_2$, then $n_1 \leq n_2$.

5. [Corr2] If $\models I \doteq I'$, then $I = I'$.

Assumption 13 (Constraint Well-formedness)
If $\Psi; \Delta; \Phi \models C$ then $\Delta \vdash C$ `wf`

Lemma 14 (Well-formedness)
If $\Psi; \Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa$ and $\Delta; \Phi \vdash \Gamma$ `wf` and $FV(\Gamma) \subseteq \Delta$, then $\Phi; \Delta \vdash \tau$ `wf` and $FV(\kappa, \tau) \subseteq \Delta$.

Proof. The proof is by induction on typing derivation of e. □

Lemma 15 (Subtyping well-formedness)
If $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\Delta; \Phi \vdash \tau$ `wf` and $FV(\tau) \subseteq \Delta$, then $\Phi; \Delta \vdash \tau'$ `wf` and $FV(\tau') \subseteq \Delta$.

Proof. The proof is by induction on subtyping derivation. □

Lemma 16 (Forced type well-formedness)
If $\Delta; \Phi \vdash \tau$ `wf` and $\mu = \mathbb{S} \vee \mathbb{C}$, then $\Phi; \Delta \vdash \tau^{\downarrow\mu}$ `wf`

Proof. The proof is by induction on well-formedness judgment. □

Our fundamental theorem relies on the assumption that the semantic interpretation of every primitive function lies in the interpretation of the function's type. This is explained below.

Assumption 17 (Soundness of primitive functions (binary))
Suppose $\zeta : B_1 \cdots B_2 \xrightarrow{\kappa} B$ and $(m, \mathtt{w_i}) \in [\![(\mathtt{B_i})^{\mu_i}]\!]_{\mathtt{v}}$, then

- $\widehat{\zeta}(\mathrm{L}(\mathtt{w_1}) \cdots \mathrm{L}(\mathtt{w_n})) = (\mathtt{f_r}, \mathtt{v_r})$
- $\widehat{\zeta}(\mathrm{R}(\mathtt{w_1}) \cdots \mathrm{R}(\mathtt{w_n})) = (\mathtt{f'_r}, \mathtt{v'_r})$
- $(m, \mathtt{merge}(v_r, v'_r)) \in [\![(B)^{\mu}]\!]_v$ where $\mu_1 \sqcup \cdots \sqcup \mu_n = \mu$
- $f'_r \leq \kappa$

We define $\mathtt{merge}(\cdot, \cdot,)$ as follows: if $v_r = v'_r$, then $\mathtt{merge}(v_r, v'_r) = \ulcorner v_r \urcorner$ else $\mathtt{merge}(v_r, v'_r) = \mathtt{new}(v_r, v'_r)$.

Assumption 18 (Soundness of primitive functions (unary))
Suppose $\zeta : B_1 \cdots B_2 \xrightarrow{\kappa} B$ and $(m, v_i) \in (\!|(B_i)^{\mu_i}|\!)_v$, then

- $\widehat{\zeta}(v_1 \cdots v_n) = (f_r, v_r)$
- $(m, v_r) \in (\!|(B)^{\mu}|\!)_v$
- $f_r \leq \kappa$

Theorem 19 (Fundamental theorem for abstract semantics)
Assume that $\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\Delta; \Phi \vdash \Gamma$ `wf` and $FV(\tau, \kappa) \subseteq \Delta$,

1. If $\epsilon = \mathbb{S}$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$, then $(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa}$.

2. If $\epsilon = \mathbb{C}$ and $(k, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$ where , then $(k, \mathcal{U}(e)) \in (\!|\sigma\tau|\!)_{\varepsilon}^{\sigma\kappa}$.

3. If $\epsilon = \mathbb{C}$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$, then $(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![\sigma\tau]\!]^{\sigma\kappa}_{\varepsilon}$.

Proof. All three statements are proved by induction on e's typing with a sub-induction on step indices for recursive functions.We show select cases of the proofs. In the proof of (1), the numbers 1–5 represent the corresponding clauses in the definition of $[\![\tau]\!]^{\kappa}_{\varepsilon}$.

Proof of statement (1):

In all subcases except values, variables and primitive functions, we show the proofs where we apply the non-trivial change propagation rule that applies whenever $\neg\texttt{stable}(\theta^{\ulcorner} e^{\urcorner})$. We don't present the case where $\texttt{stable}(\theta^{\ulcorner} e^{\urcorner})$, since it is very easy to establish using the only applicable rule **r-nochange**.

In the remainder of the rules, unless otherwise stated, we assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.

Case $\dfrac{}{\Delta; \Phi; \Gamma, x : \tau \vdash_{\mathbb{S}} x : \tau \mid 0}$ **var**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma, x : \sigma\tau]\!]$ and $\models \sigma\Phi$. TS: $(m, \theta^{\ulcorner} x^{\urcorner}) \in [\![\sigma\tau]\!]^{0}_{\varepsilon}$.

By Value Lemma (Lemma 5), STS: $(m, \theta(x)) \in [\![\sigma\tau]\!]_{v}$.

This follows from the definition of $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma, x : \sigma\tau]\!]$.

Case $\dfrac{}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \texttt{r} : (\texttt{real})^{\mathbb{S}} \mid 0}$ **real**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner} \texttt{r}^{\urcorner}) \in [\![$

$tcp\texttt{real}]\!]^{0}_{\varepsilon}$.

Since $\ulcorner \texttt{r} \urcorner = \texttt{keep}(\texttt{r})$ STS:$(m, \texttt{keep}(\texttt{r})) \in [\![(\texttt{real})^{\mathbb{S}}]\!]^{0}_{\varepsilon}$

By Value Lemma (Lemma 5), STS: $(m, \texttt{keep}(\texttt{r})) \in [\![(\texttt{real})^{\mathbb{S}}]\!]_{v}$.

This follows from the definition of $[\![(\texttt{real})^{\mathbb{S}}]\!]_{v}$ and noting that $\texttt{stable}(\texttt{keep}(\texttt{r}))$.

Case $\dfrac{\Delta; \Phi \vdash \tau \ \texttt{wf}}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \texttt{nil} : (\texttt{list}\,[0]^{0}\ \tau)^{\mathbb{S}} \mid 0}$ **nil**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner} \texttt{nil}^{\urcorner}) \in [\![(\texttt{list}\,[0]^{0}\ (\sigma\tau))^{\mathbb{S}}]\!]^{0}_{\varepsilon}$.

Since $\ulcorner \texttt{nil} \urcorner = \texttt{nil}$, by the Value Lemma (Lemma 5),

STS: $(m, \texttt{nil}) \in [\![(\texttt{list}\,[0]^{0}\ (\sigma\tau))^{\mathbb{S}}]\!]_{v}$.

This follows immediately from the definition.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e_1 : \square(\tau) \mid \kappa_1 \qquad \Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e_2 : (\texttt{list}\,[n]^{\alpha}\ \tau)^{\mathbb{S}} \mid \kappa_2}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \texttt{cons}(e_1, e_2) : (\texttt{list}\,[n+1]^{\alpha}\ \tau)^{\mathbb{S}} \mid \kappa_1 + \kappa_2}$ **cons1**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner} \texttt{cons}(e_1, e_2)^{\urcorner}) \in [\![(\texttt{list}\,[\sigma n + 1]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]^{\sigma(\kappa_1+\kappa_2)}_{\varepsilon}$.

Following the definition of $[\![\cdot]\!]^{:}_{\varepsilon}$, assume that:

$$\frac{\mathrm{L}(\theta^{\ulcorner} e_1^{\urcorner}) \Downarrow T_1, f_1 \ (\star) \qquad \mathrm{L}(\theta^{\ulcorner} e_2^{\urcorner}) \Downarrow T_2, f_2 \ (\dagger) \qquad v_i = \mathtt{V}(T_i)}{\mathtt{cons}(\mathrm{L}(\theta^{\ulcorner} e_1^{\urcorner}),\, \mathrm{L}(\theta^{\ulcorner} e_2^{\urcorner})) \Downarrow \langle \mathtt{cons}(v_1,\, v_2), \mathtt{cons}(T_1, T_2)\rangle, f_1 + f_2 + c_{cons}()}\ \textbf{cons}$$

where $f = f_1 + f_2 + c_{cons}() < m$

By IH on e_1, we get $(m, \theta^{\ulcorner} e_1^{\urcorner}) \in [\![\Box(\sigma\tau)]\!]^{\sigma\kappa_1}_{\varepsilon}$. Unrolling its definition using premise ($\star$) with $f_1 < m$, we get

a) $\langle T_1, \theta^{\ulcorner} e_1^{\urcorner}\rangle \curvearrowright \mathtt{w}'_1, \mathtt{T}'_1, \mathtt{c}'_1$

b) $\mathrm{R}(\theta^{\ulcorner} e_1^{\urcorner}) \Downarrow T'_1, f'_1$

c) $v_1 = \mathrm{L}(\mathtt{w}'_1) \ \wedge \ \mathtt{V}(\mathtt{T}'_1) = \mathtt{v}'_1 = \mathrm{R}(\mathtt{w}'_1)$

d) $c'_1 \leq \sigma\kappa_1$

e) $(m - f_1, \mathtt{w}'_1) \in [\![\Box(\sigma\tau)]\!]_{\mathtt{v}}$

By IH on e_2, we get $(m, \theta^{\ulcorner} e_2^{\urcorner}) \in [\![(\mathtt{list}\,[\sigma n]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]^{\sigma\kappa_2}_{\varepsilon}$. Unrolling its definition using premise ($\dagger$) with $f_2 < m$, we get

f) $\langle T_2, \theta^{\ulcorner} e_2^{\urcorner}\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2$

g) $\mathrm{R}(\theta^{\ulcorner} e_2^{\urcorner}) \Downarrow T'_2, f'_2$

h) $v_2 = \mathrm{L}(\mathtt{w}'_2) \ \wedge \ \mathtt{V}(\mathtt{T}'_2) = \mathtt{v}'_2 = \mathrm{R}(\mathtt{w}'_2)$

i) $c'_2 \leq \sigma\kappa_2$

j) $(m - f_2, \mathtt{w}'_2) \in [\![(\mathtt{list}\,[\sigma\mathtt{n}]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

Then,

1. Using a) and f)
$$\frac{\langle T_1, \theta^{\ulcorner} e_1^{\urcorner}\rangle \curvearrowright \mathtt{w}'_1, \mathtt{T}'_1, \mathtt{c}'_1 \qquad \langle T_2, \theta^{\ulcorner} e_2^{\urcorner}\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2 \qquad \mathtt{V}(T'_i) = v'_i}{\langle \langle \_, \mathtt{cons}(T_1, T_2)\rangle, \theta^{\ulcorner} \mathtt{cons}(e_1,\, e_2)^{\urcorner}\rangle \curvearrowright \mathtt{cons}(\mathtt{w}'_1, \mathtt{w}'_2), \langle \mathtt{cons}(\mathtt{v}'_1,\, \mathtt{v}'_2), \mathtt{cons}(\mathtt{T}'_1, \mathtt{T}'_2)\rangle, \mathtt{c}'_1 + \mathtt{c}'_2}\ \textbf{r-cons1}$$
2. Using b), g), c) and h)
$$\frac{\mathrm{R}(\theta^{\ulcorner} e_1^{\urcorner}) \Downarrow T'_1, f'_1 \qquad \mathrm{R}(\theta^{\ulcorner} e_2^{\urcorner}) \Downarrow T'_2, f'_2 \qquad \mathtt{V}(T'_i) = v'_i}{\mathtt{cons}(\mathrm{R}(\theta^{\ulcorner} e_1^{\urcorner}),\, \mathrm{L}(\theta^{\ulcorner} e_2^{\urcorner})) \Downarrow \langle \mathtt{cons}(v'_1,\, v'_2), \mathtt{cons}(T'_1, T'_2)\rangle, f'_1 + f'_2 + c_{cons}()}\ \textbf{cons}$$
3. Using c) and h), $\mathtt{cons}(v_1,\, v_2) = \mathrm{L}(\mathtt{cons}(\mathtt{w}'_1, \mathtt{w}'_2)) \ \wedge \ \mathtt{cons}(\mathtt{v}'_1,\, \mathtt{v}'_2) = \mathrm{R}(\mathtt{cons}(\mathtt{w}'_1, \mathtt{w}'_2))$
4. By using d) and i), $c'_1 + c'_2 \leq \sigma(\kappa_1 + \kappa'_2)$
5. By downward closure (Lemma 3) on e) and j) using $m - f \leq m - f_1$ and $m - f \leq m - f_2$, we get
   $(m - f, \mathtt{w}'_1) \in [\![\Box(\sigma\tau)]\!]_{\mathtt{v}}$
   $(m - f, \mathtt{w}'_2) \in [\![(\mathtt{list}\,[\sigma\mathtt{n}]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$
   $\therefore (m - f, \mathtt{cons}(\mathtt{w}'_1, \mathtt{w}'_2)) \in [\![(\mathtt{list}\,[\sigma\mathtt{n} + \mathtt{1}]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

Case $$\frac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e_1 : \tau \mid \kappa_1 \qquad \Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e_2 : (\mathtt{list}\,[n]^{\alpha-1}\ \tau)^{\mathbb{S}} \mid \kappa_2 \qquad \Delta; \Phi \models \alpha > 0}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \mathtt{cons}(e_1,\, e_2) : (\mathtt{list}\,[n+1]^{\alpha}\ \tau)^{\mathbb{S}} \mid \kappa_1 + \kappa_2}\ \textbf{cons2}$$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner}\texttt{cons}(e_1, e_2)^{\urcorner}) \in [\![(\texttt{list}\,[\sigma n + 1]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\varepsilon}^{\sigma(\kappa_1+\kappa_2)}$.

Following the definition of $[\![\cdot]\!]_{\varepsilon}^{:}$, assume that:

$$\frac{\mathrm{L}(\theta^{\ulcorner}e_1^{\urcorner}) \Downarrow T_1, f_1\ (\star) \qquad \mathrm{L}(\theta^{\ulcorner}e_2^{\urcorner}) \Downarrow T_2, f_2\ (\dagger) \qquad v_i = \mathtt{V}(T_i)}{\texttt{cons}(\mathrm{L}(\theta^{\ulcorner}e_1^{\urcorner}), \mathrm{L}(\theta^{\ulcorner}e_2^{\urcorner})) \Downarrow \langle \texttt{cons}(v_1, v_2), \texttt{cons}(T_1, T_2)\rangle f_1 + f_2 + c_{cons}()}\ \textbf{cons}$$

where $f = f_1 + f_2 + c_{cons}() < m$

By IH on e_1, we get $(m, \theta^{\ulcorner}e_1^{\urcorner}) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa_1}$. Unrolling its definition using premise $(\star)$ with $f_1 \leq f < m$, we get

a) $\langle T_1, \theta^{\ulcorner}e_1^{\urcorner}\rangle \curvearrowright \mathtt{w}_1', \mathtt{T}_1', \mathtt{c}_1'$

b) $\mathrm{R}(\theta^{\ulcorner}e_1^{\urcorner}) \Downarrow T_1', f_1'$

c) $v_1 = \mathrm{L}(\mathtt{w}_1') \ \wedge\ \mathtt{V}(\mathtt{T}_1') = \mathtt{v}_1' = \mathrm{R}(\mathtt{w}_1')$

d) $c_1' \leq \sigma\kappa_1$

e) $(m - f_1, \mathtt{w}_1') \in [\![\sigma\tau]\!]_{\mathtt{v}}$

By IH on e_2, we get $(m, \theta^{\ulcorner}e_2^{\urcorner}) \in [\![(\texttt{list}\,[\sigma n]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\varepsilon}^{\sigma\kappa_2}$. Unrolling its definition using premise $(\dagger)$ with $f_2 \leq f < m$, we get

f) $\langle T_2, \theta^{\ulcorner}e_2^{\urcorner}\rangle \curvearrowright \mathtt{w}_2', \mathtt{T}_2', \mathtt{c}_2'$

g) $\mathrm{R}(\theta^{\ulcorner}e_2^{\urcorner}) \Downarrow T_2', f_2'$

h) $v_2 = \mathrm{L}(\mathtt{w}_2') \ \wedge\ \mathtt{V}(\mathtt{T}_2') = \mathtt{v}_2' = \mathrm{R}(\mathtt{w}_2')$

i) $c_2' \leq \sigma\kappa_2$

j) $(m - f_2, \mathtt{w}_2') \in [\![(\texttt{list}\,[\sigma\mathtt{n}]^{\sigma\alpha-1}\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

Then,

1. Using a), f), c) and h)
$$\frac{\langle T_1, \theta^{\ulcorner}e_1^{\urcorner}\rangle \curvearrowright \mathtt{w}_1', \mathtt{T}_1', \mathtt{c}_1' \qquad \langle T_2, \theta^{\ulcorner}e_2^{\urcorner}\rangle \curvearrowright \mathtt{w}_2', \mathtt{T}_2', \mathtt{c}_2' \qquad \mathtt{V}(T_i') = v_i'}{\langle\langle \_, \texttt{cons}(T_1, T_2)\rangle, \theta^{\ulcorner}\texttt{cons}(e_1, e_2)^{\urcorner}\rangle \curvearrowright \texttt{cons}(\mathtt{w}_1', \mathtt{w}_2'), \langle \texttt{cons}(\mathtt{v}_1', \mathtt{v}_2'), \texttt{cons}(\mathtt{T}_1', \mathtt{T}_2')\rangle, \mathtt{c}_1' + \mathtt{c}_2'}\ \textbf{r-cons1}$$
2. Using b) and g)
$$\frac{\mathrm{R}(\theta^{\ulcorner}e_1^{\urcorner}) \Downarrow T_1', f_1' \qquad \mathrm{R}(\theta^{\ulcorner}e_2^{\urcorner}) \Downarrow T_2', f_2' \qquad \mathtt{V}(T_i') = v_i'}{\texttt{cons}(\mathrm{R}(\theta^{\ulcorner}e_1^{\urcorner}), \mathrm{L}(\theta^{\ulcorner}e_2^{\urcorner})) \Downarrow \langle \texttt{cons}(v_1', v_2'), \texttt{cons}(T_1', T_2')\rangle, f_1' + f_2' + c_{cons}()}\ \textbf{cons}$$
3. Using c) and h), $\texttt{cons}(v_1, v_2) = \mathrm{L}(\texttt{cons}(\mathtt{w}_1', \mathtt{w}_2')) \ \wedge\ \texttt{cons}(\mathtt{v}_1', \mathtt{v}_2') = \mathrm{R}(\texttt{cons}(\mathtt{w}_1', \mathtt{w}_2'))$
4. By using d) and i), $c_1' + c_2' \leq \sigma(\kappa_1 + \kappa_2')$
5. By downward closure (Lemma 3) on e) and j) using $m - f \leq m - f_1$ and $m - f \leq m - f_2$, we get
   $(m - f, \mathtt{w}_1') \in [\![\sigma\tau]\!]_{\mathtt{v}}$
   $(m - f, \mathtt{w}_2') \in [\![(\texttt{list}\,[\sigma\mathtt{n}]^{\sigma\alpha-1}\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$
   $\therefore (m - f, \texttt{cons}(\mathtt{w}_1', \mathtt{w}_2')) \in [\![(\texttt{list}\,[\sigma\mathtt{n} + \mathtt{1}]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

Case $\dfrac{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e_1 : \tau \mid \kappa_1 \qquad \Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e_2 : (\texttt{list}\,[n]^{\alpha-1}\ \tau)^{\mathbb{C}} \mid \kappa_2 \qquad \Delta;\Phi \models \alpha > 0}{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} \texttt{cons}(e_1,\, e_2) : (\texttt{list}\,[n+1]^{\alpha}\ \tau)^{\mathbb{C}} \mid \kappa_1 + \kappa_2}$ **cons2**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner}\texttt{cons}(e_1, e_2)^{\urcorner}) \in [\![(\texttt{list}\,[\sigma n + 1]^{\sigma\alpha}\ \tau)^{\mathbb{C}}]\!]_{\varepsilon}^{\sigma(\kappa_1+\kappa_2)}$.

Following the definition of $[\![\cdot]\!]_{\varepsilon}^{:}$, assume that:

$\dfrac{\mathrm{L}(\theta^{\ulcorner}e_1^{\urcorner}) \Downarrow T_1, f_1\ (\star) \qquad \mathrm{L}(\theta^{\ulcorner}e_2^{\urcorner}) \Downarrow T_2, f_2\ (\dagger) \qquad v_i = \texttt{V}(T_i)}{\texttt{cons}(\mathrm{L}(\theta^{\ulcorner}e_1^{\urcorner}),\, \mathrm{L}(\theta^{\ulcorner}e_2^{\urcorner})) \Downarrow \langle \texttt{cons}(v_1,\, v_2), \texttt{cons}(T_1, T_2)\rangle, f_1 + f_2 + c_{cons}()}$ **cons**

where $f = f_1 + f_2 + c_{cons}() < m$

By IH on e_1, we get $(m, \theta^{\ulcorner}e_1^{\urcorner}) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa_1}$. Unrolling its definition using premise ($\star$) with $f_1 \leq f < m$, we get

a) $\langle T_1, \theta^{\ulcorner}e_1^{\urcorner}\rangle \curvearrowright \texttt{w}_1', \texttt{T}_1', \texttt{c}_1'$

b) $\mathrm{R}(\theta^{\ulcorner}e_1^{\urcorner}) \Downarrow T_1', f_1'$

c) $v_1 = \mathrm{L}(\texttt{w}_1') \ \wedge\ \texttt{V}(\texttt{T}_1') = \texttt{v}_1' = \mathrm{R}(\texttt{w}_1')$

d) $c_1' \leq \sigma\kappa_1$

e) $(m - f_1, \texttt{w}_1') \in [\![\sigma\tau]\!]_{\texttt{v}}$

By IH on e_2, we get $(m, \theta^{\ulcorner}e_2^{\urcorner}) \in [\![(\texttt{list}\,[\sigma n]^{\sigma\alpha-1}\ \tau)^{\mathbb{C}}]\!]_{\varepsilon}^{\sigma\kappa_2}$. Unrolling its definition using premise ($\dagger$) with $f_2 \leq f < m$, we get

f) $\langle T_2, \theta^{\ulcorner}e_2^{\urcorner}\rangle \curvearrowright \texttt{w}_2', \texttt{T}_2', \texttt{c}_2'$

g) $\mathrm{R}(\theta^{\ulcorner}e_2^{\urcorner}) \Downarrow T_2', f_2'$

h) $v_2 = \mathrm{L}(\texttt{w}_2') \ \wedge\ \texttt{V}(\texttt{T}_2') = \texttt{v}_2' = \mathrm{R}(\texttt{w}_2')$

i) $c_2' \leq \sigma\kappa_2$

j) $(m - f_2, \texttt{w}_2') \in [\![(\texttt{list}\,[\sigma\texttt{n}]^{\sigma\alpha-1}\ \tau)^{\mathbb{C}}]\!]_{\texttt{v}}$

Then, there are two cases for j)

subcase 1: $\texttt{w}_2' = \texttt{new}(\texttt{v}_2, \texttt{v}_2')$ s.t. $(m - f_2, \texttt{new}(v_2, v_2')) \in [\![(\texttt{list}\,[\sigma n]^{\sigma\alpha-1}\ \tau)^{\mathbb{C}}]\!]_{v}$

Then we can conclude with:

1. Using a), f), c) and h)

$\dfrac{\langle T_1, \theta^{\ulcorner}e_1^{\urcorner}\rangle \curvearrowright \texttt{w}_1', \texttt{T}_1', \texttt{c}_1' \qquad \langle T_2, \theta^{\ulcorner}e_2^{\urcorner}\rangle \curvearrowright \texttt{new}(v_2, v_2'), ct_2', c_2' \qquad \texttt{V}(T_i') = v_i'}{\langle\langle \texttt{cons}(v_1,\, v_2), \texttt{cons}(T_1, T_2)\rangle, \theta^{\ulcorner}\texttt{cons}(e_1,\, e_2)^{\urcorner}\rangle \curvearrowright \texttt{new}(\texttt{cons}(v_1,\, v_2), \texttt{cons}(v_2,\, v_2'))\ \langle \texttt{cons}(v_1',\, v_2'), \texttt{cons}(T_1', T_2')\rangle, c_1' + c_2'}$ **r-cons2**

2. Using b) and g)

$\dfrac{\mathrm{R}(\theta^{\ulcorner}e_1^{\urcorner}) \Downarrow T_1', f_1' \qquad \mathrm{R}(\theta^{\ulcorner}e_2^{\urcorner}) \Downarrow T_2', f_2' \qquad \texttt{V}(T_i') = v_i'}{\texttt{cons}(\mathrm{R}(\theta^{\ulcorner}e_1^{\urcorner}),\, \mathrm{L}(\theta^{\ulcorner}e_2^{\urcorner})) \Downarrow \langle \texttt{cons}(v_1',\, v_2'), \texttt{cons}(T_1', T_2')\rangle, f_1' + f_2' + c_{cons}()}$ **cons**

3. Using c) and h), $\texttt{cons}(v_1,\, v_2) = \mathrm{L}(\texttt{new}(\texttt{cons}(\mathrm{L}(\texttt{w}_1'),\, \texttt{v}_2), \texttt{cons}(\mathrm{R}(\texttt{w}_1'),\, \texttt{v}_2'))) \wedge \texttt{cons}(\texttt{v}_1',\, \texttt{v}_2') = \mathrm{R}(\texttt{new}(\texttt{cons}(\mathrm{L}(\texttt{w}_1'),\, \texttt{v}_2), \texttt{cons}(\mathrm{R}(\texttt{w}_1'),\, \texttt{v}_2')))$

4. By using d) and i), $c_1' + c_2' \leq \sigma(\kappa_1 + \kappa_2')$
5. By downward closure (Lemma 3) on e) and j) using $m - f \leq m - f_1$ and $m - f \leq m - f_2$, we get
   $(m - f, \mathtt{w}_1') \in [\![\sigma\tau]\!]_{\mathtt{v}}$
   $(m - f, \mathtt{new}(v_2, v_2')) \in [\![(\mathtt{list}\ [\sigma n]^{\sigma\alpha - 1}\ \tau)^{\mathbb{C}}]\!]_v$
   $\therefore (m - f, \mathtt{new}(\mathtt{cons}(\mathrm{L}(\mathtt{w}_1'), \mathtt{v}_2), \mathtt{cons}(\mathrm{R}(\mathtt{w}_1'), \mathtt{v}_2'))) \in [\![(\mathtt{list}\ [\sigma\mathtt{n} + \mathtt{1}]^{\sigma\alpha}\ \tau)^{\mathbb{C}}]\!]_{\mathtt{v}}$

subcase 2: $(m - f_2, \mathtt{w}_2') \in [\![\mathtt{list}\ [\sigma\mathtt{n}]^{\sigma\alpha - 1}\ \tau]\!]_{\mathtt{v}} \subseteq [\![(\mathtt{list}\ [\sigma\mathtt{n}]^{\sigma\alpha - 1}\ \tau)^{\mathbb{C}}]\!]_{\mathtt{v}}$
Then we can conclude with

1. Using a), f), c) and h)
$$\frac{\langle T_1, \theta^{\ulcorner} e_1 {}^{\urcorner}\rangle \curvearrowright \mathtt{w}_1', \mathtt{T}_1', \mathtt{c}_1' \qquad \langle T_2, \theta^{\ulcorner} e_2 {}^{\urcorner}\rangle \curvearrowright \mathtt{w}_2', \mathtt{T}_2', \mathtt{c}_2' \qquad \mathtt{w}_2' \neq \mathtt{new}(\_,\_)}{\begin{array}{c}\langle\langle \_, \mathtt{cons}(T_1, T_2)\rangle, \theta^{\ulcorner}\mathtt{cons}(e_1,\ e_2){}^{\urcorner}\rangle \curvearrowright \\ \mathtt{cons}(\mathtt{w}_1', \mathtt{w}_2'), \langle \mathtt{cons}(\mathtt{v}_1',\ \mathtt{v}_2'), \mathtt{cons}(\mathtt{T}_1', \mathtt{T}_2')\rangle, \mathtt{c}_1' + \mathtt{c}_2'\end{array}}\ \textbf{r-cons1}$$
2. Using b) and g)
$$\frac{\mathrm{R}(\theta^{\ulcorner} e_1 {}^{\urcorner}) \Downarrow T_1', f_1' \qquad \mathrm{R}(\theta^{\ulcorner} e_2 {}^{\urcorner}) \Downarrow T_2', f_2'}{\mathtt{cons}(\mathrm{R}(\theta^{\ulcorner} e_1 {}^{\urcorner}),\ \mathrm{L}(\theta^{\ulcorner} e_2 {}^{\urcorner})) \Downarrow \langle \mathtt{cons}(v_1',\ v_2'), \mathtt{cons}(T_1', T_2')\rangle, f_1' + f_2' + c_{cons}()}\ \textbf{cons}$$
3. Using c) and h), $\mathtt{cons}(v_1,\ v_2) = \mathrm{L}(\mathtt{cons}(\mathtt{w}_1', \mathtt{w}_2'))\ \wedge\ \mathtt{cons}(\mathtt{v}_1',\ \mathtt{v}_2') = \mathrm{R}(\mathtt{cons}(\mathtt{w}_1', \mathtt{w}_2'))$
4. By using d) and i), $c_1' + c_2' \leq \sigma(\kappa_1 + \kappa_2')$
5. By downward closure (Lemma 3) on e) and j) using $m - f \leq m - f_1$ and $m - f \leq m - f_2$, we get
   $(m - f, \mathtt{w}_1') \in [\![\sigma\tau]\!]_{\mathtt{v}}$
   $(m - f, \mathtt{w}_2') \in [\![\mathtt{list}\ [\sigma\mathtt{n}]^{\sigma\alpha - 1}\ \tau]\!]_{\mathtt{v}}$
   $\therefore (m - f, \mathtt{cons}(\mathtt{w}_1', \mathtt{w}_2')) \in [\![\mathtt{list}\ [\sigma\mathtt{n} + \mathtt{1}]^{\sigma\alpha}\ \tau]\!]_{\mathtt{v}} \subseteq [\![(\mathtt{list}\ [\sigma\mathtt{n} + \mathtt{1}]^{\sigma\alpha}\ \tau)^{\mathbb{C}}]\!]_{\mathtt{v}}$

Case
$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : (\mathtt{list}\ [n]^{\alpha}\ \tau)^{\mathbb{S}} \mid \kappa_e \qquad \Delta; \Phi \wedge n \doteq 0 \wedge \alpha \doteq 0; \Gamma \vdash_{\mathbb{S}} e_1 : \tau' \mid \kappa' \\ i :: \iota, \Delta; \Phi \wedge n \doteq i + 1 \wedge \alpha \leq i; h : \square(\tau), tl : (\mathtt{list}\ [i]^{\alpha}\ \tau)^{\mathbb{S}}, \Gamma \vdash_{\mathbb{S}} e_2 : \tau' \mid \kappa' \\ i :: \iota, \beta :: \iota, \Delta; \Phi \wedge n \doteq i + 1 \wedge \beta \leq i \wedge \alpha \doteq \beta + 1; h : \tau, tl : (\mathtt{list}\ [i]^{\beta}\ \tau)^{\mathbb{S}}, \Gamma \vdash_{\mathbb{S}} e_2 : \tau' \mid \kappa' \\ \kappa = \kappa_e + \kappa'\end{array}}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \mathtt{case}_{\mathtt{L}}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2 : \tau' \mid \kappa}\ \textbf{caseL}$$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$ and $\Delta; \Phi \vdash \Gamma\ \mathtt{wf}$.
TS: $(m, \theta^{\ulcorner}(\mathtt{case}_{\mathtt{L}}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2){}^{\urcorner}) \in [\![\sigma\tau']\!]_{\varepsilon}^{\sigma\kappa}$.
Urolling the definition of $[\![\cdot]\!]_{\varepsilon}^{\cdot}$, we have two cases:

subcase 1: Assume that
$$\frac{\mathrm{L}(\theta^{\ulcorner} e {}^{\urcorner}) \Downarrow T, f_e\ (\star) \qquad \mathtt{nil} = \mathtt{V}(T) \qquad \mathrm{L}(\theta^{\ulcorner} e_1 {}^{\urcorner}) \Downarrow T_1, f_1\ (\star\star) \qquad v_1 = \mathtt{V}(T_1)}{\begin{array}{c}\mathrm{L}(\theta^{\ulcorner}\mathtt{case}_{\mathtt{L}}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2{}^{\urcorner}) \Downarrow \\ \langle v_1, \mathtt{case}_{\mathtt{nil}}(T, T_1)\rangle, f + f_1 + c_{caseL}(\mathbb{C}, \_)\end{array}}\ \textbf{case-nil}$$
where $f = f_e + f_1 + c_{caseL}(\mathbb{C}, \_) < m$

By IH 1 on e, $(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![(\texttt{list}\,[\sigma n]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\varepsilon}^{\sigma\kappa_e}$
Unrolling the definition using the premise ($\star$) with $f_e \leq f < m$ we get

a) $\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$
b) $\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f'$
c) $\mathtt{nil} = \mathrm{L}(\mathtt{w}') \ \wedge\ \mathtt{V}(\mathtt{T}') = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$ (this forces $v' = \mathtt{nil}$)
d) $c' \leq \sigma\kappa_e$
e) $(m - f_e, \mathtt{w}') \in [\![(\texttt{list}\,[\sigma\mathtt{n}]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

Note that c) forces $\mathtt{w}' = \mathtt{nil}$ and then e) forces $\sigma n \doteq 0$ and $\sigma\alpha \doteq 0$.
Hence, by IH 1 on e_1, we get $(m, \theta^{\ulcorner} e_1{}^{\urcorner}) \in [\![\sigma\tau']\!]_{\varepsilon}^{\sigma\kappa'}$. Unrolling this with the premise marked ($\star\star$) and definition $f_1 \leq f < m$, we get

f) $\langle T_1, \theta^{\ulcorner} e_1{}^{\urcorner}\rangle \curvearrowright \mathtt{w}_1', \mathtt{T}_1', \mathtt{c}_1'$
g) $\mathrm{R}(\theta^{\ulcorner} e_1{}^{\urcorner}) \Downarrow T_1', f_1'$
h) $v_1 = \mathrm{L}(\mathtt{w}_1') \ \wedge\ \mathtt{V}(\mathtt{v}_1') = \mathtt{v}_1' = \mathrm{R}(\mathtt{w}_1')$
i) $c_1' \leq \sigma\kappa'$
j) $(m - f_1, \mathtt{w}_1') \in [\![\sigma\tau']\!]_{\mathtt{v}}$

Then, we can show

1. By a) and f)
$$\frac{\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{nil}, T', c' \qquad \langle T_1, \theta^{\ulcorner} e_1{}^{\urcorner}\rangle \curvearrowright \mathtt{w}_1', \mathtt{T}_1', \mathtt{c}_1' \qquad v_1' = \mathtt{V}(T_1')}{\begin{array}{c}\langle\langle \_, \mathtt{case}_{\mathtt{nil}}(T, T_1)\rangle, \theta^{\ulcorner}\mathtt{case}_{\mathtt{L}}\ e\ \mathtt{of\ nil}\ \to\ e_1 \mid \mathtt{cons}(h, tl)\ \to\ e_2{}^{\urcorner}\rangle \curvearrowright \\ \mathtt{w}_1', \langle \mathtt{v}_1', \mathtt{case}_{\mathtt{nil}}(\mathtt{T}', \mathtt{T}_1')\rangle, \mathtt{c}' + \mathtt{c}_1'\end{array}}\ \textbf{r-case-nil}$$
2. By b), c) and g)
$$\frac{\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f' \qquad \mathtt{V}(T') = \mathtt{nil} \qquad \mathrm{R}(\theta^{\ulcorner} e_1{}^{\urcorner}) \Downarrow T_1', f_1' \qquad v_1' = \mathtt{V}(T_1')}{\begin{array}{c}\mathrm{R}(\theta^{\ulcorner}\mathtt{case}_{\mathtt{L}}\ e\ \mathtt{of\ nil}\ \to\ e_1 \mid \mathtt{cons}(h, tl)\ \to\ e_2{}^{\urcorner}) \Downarrow \\ \langle v_1', \mathtt{case}_{\mathtt{nil}}(T', T_1')\rangle f' + f_1' + c_{caseL}(\mathbb{C}, \_)\end{array}}\ \textbf{case-nil}$$
3. is immediate from h)
4. By d) and i), $c' + c_1' \leq \sigma(\kappa + \kappa')$
5. By downward-closure (Lemma 3) on j) using $m - f_e - f_1 - c_{caseL}(\mathbb{C}, \_) \leq m - f_1$, we get $(m - f_e - f_1 - c_{caseL}(\mathbb{C}, \_), \mathtt{w}_1') \equiv (\mathtt{m} - \mathtt{f}, \mathtt{w}_1') \in [\![\sigma\tau']\!]_{\mathtt{v}}$

subcase 2:
$$\frac{\mathrm{L}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow \mathtt{cons}(v_h, v_{tl}), Tf\ (\star) \qquad \mathrm{L}(\theta^{\ulcorner} e_2{}^{\urcorner})[v_h/h, v_{tl}/tl] \Downarrow v_2, T_2 f_2\ (\star\star)}{\begin{array}{c}\mathrm{L}(\theta^{\ulcorner}\mathtt{case}_{\mathtt{L}}\ e\ \mathtt{of\ nil}\ \to\ e_1 \mid \mathtt{cons}(h, tl)\ \to\ e_2{}^{\urcorner}) \Downarrow \\ \langle v_2, \mathtt{case}_{\mathtt{cons}}(T, T_2)\rangle f + f_1 + c_{caseL}(\mathbb{C}, \_)\end{array}}\ \textbf{case-cons}$$
where $j = f + f_2 + c_{caseL}(\mathbb{C}, \_) < m$
By IH 1 on e, $(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![(\texttt{list}\,[\sigma n]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\varepsilon}^{\sigma\kappa}$.
Unrolling the definition with ($\star$) and the definition $f < m$, we get

a) $\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$

b) $\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow v', T' f'$

c) $\mathtt{cons}(v_h,\, v_{tl}) = \mathrm{L}(\mathtt{w}') \ \wedge \ \mathtt{v}' = \mathrm{R}(\mathtt{w}')$

d) $c' \leq \sigma\kappa_e$

e) $(m - f, \mathtt{w}') \in [\![(\mathtt{list}\,[\sigma \mathtt{n}]^{\sigma\alpha}\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

If $\sigma n = \mathtt{0}$ then $\mathtt{w}' = \mathtt{nil}$. However, this is impossible as it contradicts c).

If $\sigma n = I + 1$ then $\mathtt{w}' = \mathtt{cons}(\mathtt{w}'_{\mathtt{h}}, \mathtt{w}'_{\mathtt{tl}})$ for some $\mathtt{w}'_{\mathtt{h}}$ and $\mathtt{w}'_{\mathtt{tl}}$. By c), $v_h = \mathrm{L}(\mathtt{w}'_{\mathtt{h}})$ and $v_{tl} = \mathrm{L}(\mathtt{w}'_{\mathtt{tl}})$. Let $v'_h = \mathrm{R}(\mathtt{w}'_{\mathtt{h}})$ and $v'_{tl} = \mathrm{R}(\mathtt{w}'_{\mathtt{tl}})$.

Now, $(m - f, \mathtt{cons}(\mathtt{w}'_{\mathtt{h}}, \mathtt{w}'_{\mathtt{tl}})) \in [\![(\mathtt{list}\,[\mathtt{I}+\mathtt{1}]^{\sigma\alpha}\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$ may hold in one of two ways:

case 1. $(m - f, \mathtt{w}'_{\mathtt{h}}) \in [\![\Box(\sigma\tau)]\!]_{\mathtt{v}}$ and $(m - f, \mathtt{w}'_{\mathtt{tl}}) \in [\![(\mathtt{list}\,[\mathtt{I}]^{\sigma\alpha}\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

case 2. $(m - f, \mathtt{w}'_{\mathtt{h}}) \in [\![\sigma\tau]\!]_{\mathtt{v}}$ and $(m - f, \mathtt{w}'_{\mathtt{tl}}) \in [\![(\mathtt{list}\,[\mathtt{I}]^{\sigma\alpha-1}\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

We analyze these cases separately:

case 1. $(m - f, \mathtt{w}'_{\mathtt{h}}) \in [\![\Box(\sigma\tau)]\!]_{\mathtt{v}}$ and $(m - f, \mathtt{w}'_{\mathtt{tl}}) \in [\![(\mathtt{list}\,[\mathtt{I}]^{\sigma\alpha}\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

By IH 1 on e_2 (the third premise of the typing rule) using

- $\sigma[i \mapsto I] \in \mathcal{D}[\![i :: \iota, \Delta]\!]$ and $\sigma[i \mapsto I]\epsilon = \mathbb{S}$ since $i \notin FV(\epsilon)$.
- $\models \sigma[i \mapsto I](\Phi \wedge n \doteq i + 1 \wedge \alpha \leq i)$ since $\sigma n = I + 1$ by e) and $\sigma\alpha \leq I$ by Lemma 10.
- $(m - f, \theta[h \mapsto \mathtt{w}'_{\mathtt{h}}, \mathtt{tl} \mapsto \mathtt{w}'_{\mathtt{tl}}]) \in \mathcal{G}[\![\sigma[\mathtt{i} \mapsto \mathtt{I}](\Gamma, \mathtt{h} : \Box(\tau), \mathtt{tl} : (\mathtt{list}\,[\mathtt{i}]^{\alpha}\ \tau)^{\mathbb{S}})]\!]$, which holds because
  - $(m-f, \theta) \in \mathcal{G}[\![\sigma[i \mapsto I]\Gamma]\!] = \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ and $m - f \leq m$, and noting that $i \notin FV(\Gamma)$
  - $(m - f, \mathtt{w}'_{\mathtt{h}}) \in [\![\Box(\sigma[\mathtt{i} \mapsto \mathtt{I}]\tau)]\!]_{\mathtt{v}}$ by the assumption of e) and noting that $i \notin FV(\tau)$
  - $(m - f, \mathtt{w}'_{\mathtt{tl}}) \in [\![(\mathtt{list}\,[\mathtt{I}]^{(\sigma[\mathtt{i}\mapsto\mathtt{I}])\alpha}\ (\sigma[\mathtt{i} \mapsto \mathtt{I}])\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$ by the assumption of e) and noting that $i \notin FV(\tau, \alpha)$

Then, we get $(m, \theta[\mathtt{w}'_{\mathtt{h}}/\mathtt{h}, \mathtt{w}'_{\mathtt{tl}}/\mathtt{tl}]^{\ulcorner} \mathtt{e}_2 {}^{\urcorner}) \in [\![\sigma\tau']\!]^{\sigma\kappa'}_{\varepsilon}$, by noting that $i, \notin FV(\tau', \kappa')$.

Unrolling this definition with the premise marked ($\star\star$) and the definition $f_2 < m - f$, we get

f) $\langle T_2, \theta^{\ulcorner} e_2 {}^{\urcorner}[\mathtt{w}'_{\mathtt{h}}/\mathtt{h}, \mathtt{w}'_{\mathtt{tl}}/\mathtt{tl}]\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2$

g) $\mathrm{R}(\theta^{\ulcorner} e_2 {}^{\urcorner}[\mathtt{w}'_{\mathtt{h}}/\mathtt{h}, \mathtt{w}'_{\mathtt{tl}}/\mathtt{tl}]) \Downarrow \mathtt{T}'_2, \mathtt{f}'_2$ or, alternatively, $\mathrm{R}(\theta^{\ulcorner} e_2 {}^{\urcorner})[v'_h/h, v'_{tl}/tl] \Downarrow T'_2, f'_2$ since by e) $\mathrm{R}(\mathtt{w}'_{\mathtt{h}}) = \mathtt{v}'_{\mathtt{h}}$ and $\mathrm{R}(\mathtt{w}'_{\mathtt{tl}}) = \mathtt{v}'_{\mathtt{tl}}$

h) $v_2 = \mathrm{L}(\mathtt{w}'_2) \ \wedge \ \mathtt{V}(\mathtt{T}'_2) = \mathtt{v}'_2 = \mathrm{R}(\mathtt{w}'_2)$

i) $c'_2 \leq \sigma\kappa'$

j) $(m - f - f_2, \mathtt{w}'_2) \in [\![\sigma\tau']\!]_{\mathtt{v}}$

1. By a), e) and f)

$$\frac{\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{cons}(\mathtt{w}'_{\mathtt{h}}, \mathtt{w}'_{\mathtt{tl}}), \mathtt{T}', \mathtt{c}' \qquad \langle T_2, \theta^{\ulcorner} e_2 {}^{\urcorner}[\mathtt{w}'_{\mathtt{h}}/\mathtt{h}, \mathtt{w}'_{\mathtt{tl}}/\mathtt{tl}]\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2 \qquad \mathtt{V}(T'_2) = v'_2}{\langle\langle \_, \mathtt{case}_{\mathtt{cons}}(T, T_1)\rangle, \theta^{\ulcorner} \mathtt{case}_{\mathtt{L}}\ e\ \mathtt{of\ nil} \ \to \ e_1 \mid \mathtt{cons}(h, tl) \ \to \ e_2 {}^{\urcorner}\rangle \curvearrowright \mathtt{w}'_2, \langle \mathtt{v}'_2, \mathtt{case}_{\mathtt{cons}}(\mathtt{T}', \mathtt{T}'_2)\rangle, \mathtt{c}' + \mathtt{c}'_2}\ \textbf{r-case-cons}$$

2. By b) and g)

$$\frac{\mathrm{V}(T') = \mathtt{cons}(v'_h,\, v'_{tl}) \qquad \mathrm{R}(\theta^{\ulcorner} e_2 {}^{\urcorner})[v'_h/h, v'_{tl}/tl] \Downarrow T'_2, f'_2 \qquad \mathtt{V}(T'_2) = v'_2}{\begin{array}{c}\mathrm{R}(\theta^{\ulcorner}\mathtt{case_L}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h,tl)\ \rightarrow\ e_2{}^{\urcorner}) \Downarrow \\ \langle v'_2, \mathtt{case_{cons}}(T', T'_2)\rangle, f' + f'_2 + c_{caseL}(\mathbb{C}, \_)\end{array}}\ \textbf{case-cons}$$

3. is immediate from h)
4. By d) and i) $c' + c'_2 \leq \sigma(\kappa + \kappa')$
5. By downward-closure (Lemma 3) on j) using $m - f - f_2 - c_{caseL}(\mathbb{C}, \_) \leq m - f - f_2$, we get $(m - f - f_2 - c_{caseL}(\mathbb{C}, \_), \mathtt{w}'_2) \equiv (\mathtt{m} - \mathtt{f}, \mathtt{w}'_2) \in [\![\sigma\tau']\!]_{\mathtt{v}}$

case 2. $(m - f, \mathtt{w}'_{\mathtt{h}}) \in [\![\sigma\tau]\!]_{\mathtt{v}}$ and $(m - f, \mathtt{w}'_{\mathtt{tl}}) \in [\![(\mathtt{list}\,[\mathtt{I}]^{\sigma\alpha-1}\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$

By IH 1 on e_2 (the fourth premise of the typing rule) using

- $\sigma[i \mapsto I, \beta \mapsto \alpha - 1] \in \mathcal{D}[\![i :: \iota, \beta :: \iota, \Delta]\!]$
- $\models \sigma[i \mapsto I, \beta \mapsto \alpha - 1](\Phi \wedge n \doteq i + 1 \wedge \alpha \doteq \beta + 1 \wedge \beta \leq i)$ since $n = I + 1$ by e) and $\sigma\alpha \leq I + 1$ by Lemma 9.
- $(m - f, \theta[h \mapsto \mathtt{w}'_{\mathtt{h}}, \mathtt{tl} \mapsto \mathtt{w}'_{\mathtt{tl}}]) \in \mathcal{G}[\![\sigma[\mathtt{i} \mapsto \mathtt{I}, \beta \mapsto \alpha - \mathtt{1}](\Gamma, \mathtt{h} : \tau, \mathtt{tl} : (\mathtt{list}\,[\mathtt{i}]^{\beta}\ \tau)^{\mathbb{S}})]\!]$, which holds because
  - $(m - f, \theta) \in \mathcal{G}[\![\sigma[i \mapsto I, \beta \mapsto \alpha - 1]\Gamma]\!] = \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ and $m - f \leq m$, and noting that $i, \beta \notin FV(\Gamma)$
  - $(m - f, \mathtt{w}'_{\mathtt{h}}) \in [\![\sigma[\mathtt{i} \mapsto \mathtt{I}, \beta \mapsto \alpha - \mathtt{1}]\tau]\!]_{\mathtt{v}}$ by the assumption of e) and noting that $i, \beta \notin FV(\tau)$
  - $(m - f, \mathtt{w}'_{\mathtt{tl}}) \in [\![(\mathtt{list}\,[\mathtt{I}]^{(\sigma[\mathtt{i}\mapsto\mathtt{I},\beta\mapsto\alpha-1])\alpha-1}\ (\sigma[\mathtt{i} \mapsto \mathtt{I}, \beta \mapsto \alpha - \mathtt{1}])\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$ by the assumption of e) and noting that $i, \beta \notin FV(\tau, \alpha)$

Then, we get $(m, \theta[\mathtt{w}'_{\mathtt{h}}/\mathtt{h}, \mathtt{w}'_{\mathtt{tl}}/\mathtt{tl}]^{\ulcorner}\mathtt{e}_2{}^{\urcorner}) \in [\![\sigma\tau']\!]^{\sigma\kappa'}_{\varepsilon}$, by noting that $i, \beta \notin FV(\tau', \kappa')$.

Unrolling its definition the premise marked and the definition $j_2 = f_2 < m - f$, we get

f) $\langle T_2, \theta^{\ulcorner} e_2 {}^{\urcorner}[\mathtt{w}'_{\mathtt{h}}/\mathtt{h}, \mathtt{w}'_{\mathtt{tl}}/\mathtt{tl}]\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2$

g) $\mathrm{R}(\theta^{\ulcorner} e_2 {}^{\urcorner}[\mathtt{w}'_{\mathtt{h}}/\mathtt{h}, \mathtt{w}'_{\mathtt{tl}}/\mathtt{tl}]) \Downarrow \mathtt{T}'_2, \mathtt{f}'_2$ or, alternatively, $\mathrm{R}(\theta^{\ulcorner} e_2 {}^{\urcorner})[v'_h/h, v'_{tl}/tl] \Downarrow T'_2, f'_2$ since $\mathrm{R}(\mathtt{w}'_{\mathtt{h}}) = \mathtt{v}'_{\mathtt{h}}$ and $\mathrm{R}(\mathtt{w}'_{\mathtt{tl}}) = \mathtt{v}'_{\mathtt{tl}}$ by e)

h) $v_2 = \mathrm{L}(\mathtt{w}'_2) \ \wedge\ \mathtt{V}(\mathtt{T}'_2) = \mathtt{v}'_2 = \mathrm{R}(\mathtt{w}'_2)$

i) $c'_2 \leq \sigma\kappa'$

j) $(m - f - f_2, \mathtt{w}'_2) \in [\![\sigma\tau']\!]_{\mathtt{v}}$

1. By a), e) and f)

$$\frac{\mathtt{V}(T') = \mathtt{cons}(\mathtt{w}'_{\mathtt{h}}, \mathtt{w}'_{\mathtt{tl}}) \qquad \langle T_2, \theta^{\ulcorner} e_2 {}^{\urcorner}[\mathtt{w}'_{\mathtt{h}}/\mathtt{h}, \mathtt{w}'_{\mathtt{tl}}/\mathtt{tl}]\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2 \qquad \mathtt{V}(T'_2) = v'_2}{\begin{array}{c}\langle\langle \_, \mathtt{case_{cons}}(T, T_1)\rangle, \theta^{\ulcorner}\mathtt{case_L}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h,tl)\ \rightarrow\ e_2{}^{\urcorner}\rangle \curvearrowright \\ \mathtt{w}'_2, \langle \mathtt{v}'_2, \mathtt{case_{cons}}(\mathtt{T}', \mathtt{T}'_2)\rangle, \mathtt{c}' + \mathtt{c}'_2\end{array}}\ \textbf{r-case-cons}$$

2. By b) and g)

$$\frac{\begin{array}{c}\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f' \\ \mathtt{V}(T') = \mathtt{cons}(v'_h,\, v'_{tl}) \qquad \mathrm{R}(\theta^{\ulcorner} e_2{}^{\urcorner})[v'_h/h, v'_{tl}/tl] \Downarrow v'_2, T'_2 f'_2 \qquad \mathtt{V}(T'_2) = v'_2\end{array}}{\begin{array}{c}\mathrm{R}(\theta^{\ulcorner} \mathtt{case}_{\mathtt{L}}\ e\ \mathtt{of\ nil}\ \to\ e_1 \mid \mathtt{cons}(h, tl)\ \to\ e_2{}^{\urcorner}) \Downarrow \\ \langle v'_2, \mathtt{case}_{\mathtt{cons}}(T', T'_2)\rangle, f' + f'_2 + c_{case}(\mathbb{C}, \_)\end{array}}\ \textbf{case-cons}$$

3. follows immediately from h)
4. By d) and i) $c' + c'_2 \leq \sigma(\kappa + \kappa')$
5. By downward-closure (Lemma 3) on j) using $m - f - f_2 - c_{caseL}(\mathbb{C}, \_) \leq m - f - f_2$, we get $(m - f - f_2 - c_{caseL}(\mathbb{C}, \_)\mathtt{w}'_2) \in [\![\sigma\tau']\!]_{\mathtt{v}}$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : \tau_1 \mid \kappa}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \mathtt{inl}\ e : (\tau_1 + \tau_2)^{\mathbb{S}} \mid \kappa}\ \textbf{inl}$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner} \mathtt{inl}\ e^{\urcorner}) \in [\![(\sigma\tau_1 + \sigma\tau_2)^{\mathbb{S}}]\!]^{\kappa}_{\varepsilon}$.

Following the definition of $[\![\cdot]\!]^{\cdot}_{\varepsilon}$, assume that

$$\frac{\mathrm{L}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T, f\ (\star) \qquad \mathtt{V}(T) = v}{\mathrm{L}(\theta^{\ulcorner} \mathtt{inl}\ e^{\urcorner}) \Downarrow \langle \mathtt{inl}\ v, \mathtt{inl}\ T\rangle, f + c_{inl}()}\ \textbf{inl}$$

where $f + c_{inl}() < m$

By IH on e, $(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![\sigma\tau_1]\!]^{\sigma\kappa}_{\varepsilon}$.

Unrolling this, using the premise marked $\star$ with the definition $f < m$, we get

a) $\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$

b) $\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f'$

c) $v = \mathrm{L}(\mathtt{w}') \ \wedge\ \mathtt{V}(\mathtt{T}') = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$

d) $c' \leq \sigma\kappa$

e) $(m - f, \mathtt{w}') \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$

Now, we can show:

1. By a),
$$\frac{\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{w}, \mathtt{T}', \mathtt{c}' \qquad \mathtt{V}(T') = v'}{\langle\langle \_, \mathtt{inl}\ T\rangle, \theta^{\ulcorner} \mathtt{inl}\ e^{\urcorner}\rangle \curvearrowright \mathtt{inl}\ \mathtt{w}, \langle \mathtt{inl}\ \mathtt{v}', \mathtt{inl}\ \mathtt{T}'\rangle, \mathtt{c}'}\ \textbf{r-inl}$$
2. By b),
$$\frac{\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f' \qquad \mathtt{V}(T') = v'}{\mathrm{R}(\theta^{\ulcorner} \mathtt{inl}\ e^{\urcorner}) \Downarrow \langle \mathtt{inl}\ v', \mathtt{inl}\ T'\rangle, f' + c_{inl}()}\ \textbf{inl}$$
3. By c), $\mathtt{inl}\ v = \mathrm{L}(\mathtt{inl}\ \mathtt{w}') \ \wedge\ \mathtt{inl}\ \mathtt{v}' = \mathrm{R}(\mathtt{inl}\ \mathtt{w}')$
4. By d)
5. TS: $(m - f - c_{inl}(), \mathtt{inl}\ \mathtt{w}') \in [\![(\sigma\tau_1 + \sigma\tau_2)^{\mathbb{S}}]\!]_{\mathtt{v}}$.
   By unrolling the definition, STS: $(m - f - c_{inl}(), \mathtt{w}') \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$. This follows by Lemma 3 using e) and $m - f - c_{inl}() \leq m - f$.

Case $\dfrac{\Delta; \Phi; f : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}, x : \tau_1, \Gamma \vdash_\delta e : \tau_2 \mid \kappa}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \texttt{fix}\ f(x).e : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}} \mid 0}$ **fix1**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.
TS: $(m, \theta^{\ulcorner}\texttt{fix}\ f(x).e^{\urcorner}) \in [\![(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]^0_\varepsilon$.
STS: $(m, \theta^{\ulcorner}\texttt{fix}\ f(x).e^{\urcorner}) \in [\![(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 5. Let $F = \theta^{\ulcorner}\texttt{fix}\ f(x).e^{\urcorner}$.
There are two cases.

subcase 1: $\delta = \mathbb{S}$

We prove the more general statement
$\forall k \le m.\ (k, \theta^{\ulcorner}\texttt{fix}\ f(x).e^{\urcorner}) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by subinduction on k.

subsubcase 1: $k = 0$ is vacuous from the definition $[\![\cdot]\!]_v$ at the function type.

subsubcase 2: $k + 1 \le m$

Assume, by the sub-IH, that $(k, F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ ($\star$)
STS: $(k+1, F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$
Following the definition, pick $j < k + 1$.
Assume that $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$. Then, STS: $(j, \theta^{\ulcorner}e^{\urcorner}[F/f, \mathtt{w}/\mathtt{x}]) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_\varepsilon$ ($\star\star$).
Instantiate the IH 1 on the premise of the typing rule using:
$(j, \theta[f \mapsto F, x \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma(\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : (\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \tau_2)^{\mathbb{S}})]\!]$, which holds because:

* $(j, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ and $j \le m$,
* $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$
* $(j, F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on ($\star$) and $j \le k$

We immediately get $(j, \theta[f \mapsto F, x \mapsto \mathtt{w}]^{\ulcorner}\mathtt{e}^{\urcorner}) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_\varepsilon$, which is same as ($\star\star$).

subcase 2: $\delta = \mathbb{C}$

Remember that $F = \theta^{\ulcorner}\texttt{fix}\ f(x).e^{\urcorner}$.

- **STS 1:** $\forall k.\ (k, \mathrm{L}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v \ \wedge\ (k, \mathrm{R}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$.
  Proof proceeds by sub-induction on k.
  i. **case** $k = 0$ is vacuous from the definition of $[\![\cdot]\!]_v$ at the function type with C body.
  ii. **case** $k + 1$
  Assume by sub-IH that
  $(k, \mathrm{L}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$($\dagger$) and $(k, \mathrm{R}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$($\dagger\dagger$) hold.
  STS: $(k+1, \mathrm{L}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$ and $(k+1, \mathrm{R}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$
  Pick $j < k + 1$ s.t. $(j, v) \in (\!|\sigma\tau_1|\!)_v$($\dagger\dagger\dagger$). Then,
  STS1: $(j, \mathrm{L}(\theta^{\ulcorner}e^{\urcorner})[\mathrm{L}(F)/f, v/x]) \in (\!|\sigma\tau_2|\!)^{\sigma\kappa}_\varepsilon$(♠)
  STS2: $(j, \mathrm{R}(\theta^{\ulcorner}e^{\urcorner})[\mathrm{R}(F)/f, v/x]) \in (\!|\sigma\tau_2|\!)^{\sigma\kappa}_\varepsilon$(♠♠)
  We first show the first one.

Instantiate the IH 2 on the premise of the typing rule using $\delta = \mathbb{C}$ and $(j, \mathrm{L}(\theta)^{\ulcorner} e^{\urcorner}[f \mapsto \mathrm{L}(F), x \mapsto v]) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau_1, f : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)$, which holds because:

* $(j, \mathrm{L}(\theta)) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by instantiating Lemma 2 with j using $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$,
* $(j, v) \in (\!|\sigma\tau_1|\!)_v$ by (†††)
* $(j, \mathrm{L}(F)) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on (†) using $j \leq k$

We immediately get $(j, \mathrm{L}(\theta)[f \mapsto \mathrm{L}(F), x \mapsto v]\mathrm{L}(^{\ulcorner} e^{\urcorner})) \in (\!|\sigma\tau_2|\!)_{\varepsilon}^{\sigma\kappa}$, which is same as (♠).

Instantiate the IH 2 on the premise of the typing rule using $\delta = \mathbb{C}$ and $(j, \mathrm{R}(\theta)^{\ulcorner} e^{\urcorner}[f \mapsto \mathrm{R}(F), x \mapsto v]) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau_1, f : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)$, which holds because:

* $(j, \mathrm{R}(\theta)) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by instantiating Lemma 2 with j using $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$,
* $(j, v) \in (\!|\sigma\tau_1|\!)_v$ by (†††)
* $(j, \mathrm{R}(F)) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on (††) using $j \leq k$

We immediately get $(j, \mathrm{R}(\theta)[f \mapsto \mathrm{R}(F), x \mapsto v]\mathrm{R}(^{\ulcorner} e^{\urcorner})) \in (\!|\sigma\tau_2|\!)_{\varepsilon}^{\sigma\kappa}$, which is same as (♠♠).

- **STS 2:** $\forall j < m.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, {}^{\ulcorner}\mathtt{e}^{\urcorner}[\mathtt{F/f}][\mathtt{w/x}]) \in [\![\sigma\tau_2]\!]_{\varepsilon}^{\sigma\kappa}$

  Proof by sub-induction on m.

  i. **case** $m = 0$ is vacuous since there exists no positive $\mathrm{j} < 0$.

  ii. **case** $m = m' + 1$

  STS: $\forall j < m' + 1.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta^{\ulcorner}\mathtt{e}^{\urcorner}[\mathtt{F/f}][\mathtt{w/x}]) \in [\![\sigma\tau_2]\!]_{\varepsilon}^{\sigma\kappa}$.

  There are two possible cases.

  – $j < m'$

  Then, by sub-IH, we know that $\forall j < m'.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta^{\ulcorner}\mathtt{e}^{\urcorner}[\mathtt{F/f}][\mathtt{w/x}]) \in [\![\tau_2]\!]_{\varepsilon}^{\sigma\kappa}$. Since $j < m' < m' + 1$, we can immediately conclude.

  – $j = m'$

  Since $j = m' < m' + 1$, we assume that $(m', \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}(\diamond)$.

  STS: $(m', \theta^{\ulcorner} e^{\urcorner}[F/f][\mathtt{w/x}]) \in [\![\sigma\tau_2]\!]_{\varepsilon}^{\sigma\kappa}(\diamond\diamond)$

  By IH 3 on the premise of the typing rule using $(m', \theta^{\ulcorner} e^{\urcorner}[f \mapsto F, x \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma\Gamma, \mathtt{x} : \sigma\tau_1, \mathtt{f} : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]$, which holds because:

  * $(m', \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by instantiating Lemma 3 with $m' < m$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$
  * $(m', \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$ by $(\diamond)$
  * $(m', F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ since as shown in (STS 1) above, $\forall k.\ (k, F) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$ and by sub-IH, $\forall j < m'.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta^{\ulcorner}\mathtt{e}^{\urcorner}[\mathtt{F/f}][\mathtt{w/x}]) \in [\![\tau_2]\!]_{\varepsilon}^{\kappa}$.

  We immediately get $(m', \theta[f \mapsto F, x \mapsto \mathtt{w}]^{\ulcorner}\mathtt{e}^{\urcorner}) \in [\![\sigma\tau_2]\!]_{\varepsilon}^{\sigma\kappa}$, which is same as $(\diamond\diamond)$.

Case

$$\frac{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e_1 : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mu} \mid \kappa_1 \qquad \Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e_2 : \tau_1 \mid \kappa_2 \qquad \Delta;\Phi \models \mu \trianglelefteq \tau_2 \qquad \Delta;\Phi \models (\epsilon \sqcup \mu) \leq \delta \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{app}(\epsilon,\mu)\ :\ 0)}{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e_1\ e_2 : \tau_2 \mid \kappa}\ \textbf{app}$$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner} e_1\ e_2{}^{\urcorner}) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$.

Following the definition of $[\![\cdot]\!]^{\cdot}_{\varepsilon}$, assume that

$$\frac{\mathrm{L}(\theta^{\ulcorner} e_1{}^{\urcorner}) \Downarrow T_1, f_1\ (\star) \qquad \texttt{fix}\ f(x).\, e = \mathtt{V}(T_1) \qquad \mathrm{L}(\theta^{\ulcorner} e_2{}^{\urcorner}) \Downarrow T_2, f_2\ (\star\star) \qquad v_2 = \mathtt{V}(T_2) \qquad e[v_2/x, (\texttt{fix}\ f(x).\, e)/f] \Downarrow T_r, f_r\ (\star\star\star) \qquad v_r = \mathtt{V}(T_r)}{\mathrm{L}(\theta^{\ulcorner} e_1\ e_2{}^{\urcorner}) \Downarrow \langle v_r, \mathtt{app}(T_1, T_2, T_r)\rangle, f_1 + f_2 + f_r + c_{app}(\mathbb{C},\mathbb{S})}\ \textbf{app}$$

where $f = f_1 + f_2 + f_r + c_{app}(\mathbb{C},\mathbb{S}) < m$

By IH on e_1, we get $(m, \theta^{\ulcorner} e_1{}^{\urcorner}) \in [\![(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa')} \sigma\tau_2)^{\mu}]\!]^{\sigma\kappa_1}_{\varepsilon}$.

Unrolling its definition using the premise marked ($\star$) and the definition $f_1 < f < m$, we get

a) $\langle T_1, \theta^{\ulcorner} e_1{}^{\urcorner}\rangle \curvearrowright \mathtt{w}'_1, \mathtt{T}'_1, \mathtt{c}'_1$

b) $\mathrm{R}(\theta^{\ulcorner} e_1{}^{\urcorner}) \Downarrow T'_1, f'_1$

c) $\texttt{fix}\ f(x).\, e = \mathrm{L}(\mathtt{w}'_1) \ \wedge\ \mathtt{V}(\mathtt{T}'_1) = \mathtt{v}'_1 = \mathrm{R}(\mathtt{w}'_1)$

d) $c'_1 \leq \sigma\kappa_1$

e) $(m - f_1, \mathtt{w}'_1) \in [\![(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa')} \sigma\tau_2)^{\mu}]\!]_{\mathtt{v}}$

By IH on e_2 , we get $(m, \theta^{\ulcorner} e_2{}^{\urcorner}) \in [\![\sigma\tau_1]\!]^{\sigma\kappa_2}_{\varepsilon}$

Unrolling its definition, using the premise ($\star\star$) and the definition $f_2 < f < m$, we get

f) $\langle T_2, \theta^{\ulcorner} e_2{}^{\urcorner}\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2$

g) $\mathrm{R}(\theta^{\ulcorner} e_2{}^{\urcorner}) \Downarrow T'_2, f'_2$

h) $v_2 = \mathrm{L}(\mathtt{w}'_2) \ \wedge\ \mathtt{V}(\mathtt{T}'_2) = \mathtt{v}'_2 = \mathrm{R}(\mathtt{w}'_2)$

i) $c'_2 \leq \sigma\kappa_2$

j) $(m - f_2, \mathtt{w}'_2) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$

There are three cases for e).

subcase 1: $\mu = \mathbb{S}$ and $\delta = \mathbb{S}$

By e), we have $(m - f_1, \texttt{fix}\ f(x).\, \mathtt{ee}) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa')} \sigma\tau_2)^{\mathbb{S}}]\!]_{\mathtt{v}}$ ($\dagger$) for some $\mathtt{ee}$, such that by c), $v'_1 = \texttt{fix}\ f(x).\, e'$ for some $e' = \mathrm{R}(\mathtt{ee})$.

Next, we unroll ($\dagger$) with $(m - f_1 - f_2 - c_{app}(\mathbb{C},\mathbb{S}), \mathtt{w}'_2) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$ (downward-closure with Lemma 3 on j) since $m - f_1 - f_2 - c_{app}(\mathbb{C},\mathbb{S}) < m - f_2$) using

$m - f_1 - f_2 - c_{app}(\mathbb{C},\mathbb{S}) < m - f_1$, we get

$(m - f_1 - f_2 - c_{app}(\mathbb{C},\mathbb{S}), \mathtt{ee}[(\texttt{fix}\ \mathtt{f(x).\, ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}]) \in [\![\sigma\tau_2]\!]_{\mathtt{v}}$.

Unrolling its definition with the premise ($\star\star\star$), noting that

$\mathrm{L}(\mathtt{ee}[(\texttt{fix}\ \mathtt{f(x).\, ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}]) = \mathtt{e}[\mathtt{v}_2/\mathtt{x}, (\texttt{fix}\ \mathtt{f(x).\, e})/\mathtt{f}]$ from c) and h), and

$f_r < m - f_1 - f_2 - f_r - c_{app}(\mathbb{C}, \mathbb{S})$ since $j < m$, we get

k) $\langle T_r, \mathtt{ee}[(\mathtt{fix\ f(x).\ ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}]\rangle \curvearrowright \mathtt{w}'_\mathtt{r}, \mathtt{T}'_\mathtt{r}, \mathtt{c}'_\mathtt{r}$

l) $\mathrm{R}(\mathtt{ee}[(\mathtt{fix\ f(x).\ ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}] = \mathtt{e}'[(\mathtt{fix\ f(x).\ e}')/\mathtt{f}, \mathtt{v}'_2/\mathtt{x}]) \Downarrow \mathtt{T}'_\mathtt{r}, \mathtt{f}'_\mathtt{r}$

m) $v_r = \mathrm{L}(\mathtt{w}'_\mathtt{r}) \ \wedge\ \mathtt{V}(\mathtt{T}'_\mathtt{r}) = \mathtt{v}'_\mathtt{r} = \mathrm{R}(\mathtt{w}'_\mathtt{r})$

n) $c'_r \leq \sigma\kappa'$

o) $(m - f_1 - f_2 - f_r - c_{app}(\mathbb{C}, \mathbb{S}), \mathtt{w}'_\mathtt{r}) \in [\![\sigma\tau_2]\!]_\mathtt{v}$

1. By a), f) and k)

$$\frac{\langle T_1, \theta^\ulcorner e_1 \urcorner\rangle \curvearrowright \mathtt{fix}\ f(x).\mathtt{ee}, \mathtt{T}'_1, \mathtt{c}'_1 \qquad \langle T_2, \theta^\ulcorner e_2 \urcorner\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2 \qquad \langle T_r, \mathtt{ee}[(\mathtt{fix\ f(x).\ ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}]\rangle \curvearrowright \mathtt{w}'_\mathtt{r}, \mathtt{T}'_\mathtt{r}, \mathtt{c}'_\mathtt{r} \qquad v'_r = \mathtt{V}(T'_r)}{\langle\langle \_, \mathtt{app}(T_1, T_2, T_r)\rangle, \theta^\ulcorner e_1\ e_2 \urcorner\rangle \curvearrowright \mathtt{w}'_\mathtt{r}, \langle \mathtt{v}'_\mathtt{r}, \mathtt{app}(\mathtt{T}'_1, \mathtt{T}'_2, \mathtt{T}'_\mathtt{r})\rangle, \mathtt{c}'_1 + \mathtt{c}'_2 + \mathtt{c}'_\mathtt{r}}\ \textbf{r-app1}$$

2. By b), g) and l)

$$\frac{\mathrm{R}(\theta^\ulcorner e_1 \urcorner) \Downarrow T'_1, f'_1 \qquad \mathtt{fix}\ f(x).\ e' = \mathtt{V}(T'_1) \qquad \mathrm{R}(\theta^\ulcorner e_2 \urcorner) \Downarrow T'_2, f'_2 \qquad v'_2 = \mathtt{V}(T'_2) \qquad e'[(\mathtt{fix}\ f(x).\ e')/f, v'_2/x] \Downarrow T'_r, f'_r \qquad v'_r = \mathtt{V}(T'_r)}{\mathrm{R}(\theta^\ulcorner e_1\ e_2 \urcorner) \Downarrow \langle v'_r, \mathtt{app}(T'_1, T'_2, T'_r)\rangle, f'_1 + f'_2 + f'_r + c_{app}(\mathbb{C}, \mathbb{S})}\ \textbf{app}$$

3. follows immediately from m)

4. By d), i), n) $c_1 + c_2 + c_r \leq \sigma(\kappa_1 + \kappa_2 + \kappa')$

5. Since $j = f_1 + f_2 + f_r + c_{app}(\mathbb{C}, \mathbb{S})$, by o), we get $(m - j, \mathtt{w}'_\mathtt{r}) \in [\![\sigma\tau_2]\!]_\mathtt{v}$, noting that $m - j = m - f_1 - f_2 - f_r - c_{app}(\mathbb{C}, \mathbb{S})$.

Then, we conclude this subcase by showing

subcase 2: $\mu = \mathbb{S}$ and $\delta = \mathbb{C}$

The proof of this case is similar to the case where $\mu = \epsilon = \mathbb{S}$, but we will show it for clarity.
By e), we know that $(m - f_1, \mathtt{fix}\ f(x).\ \mathtt{ee}) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2)^\mathbb{S}]\!]_\mathtt{v}(\dagger)$.
By j), we know that $(m - f_2, \mathtt{w}'_2) \in [\![\sigma\tau_1]\!]_\mathtt{v}(\dagger\dagger)$.
Unrolling **the second part of** ($\dagger$) with $(m - f_1 - f_2 - c_{app}(\mathbb{C}, \mathbb{S}), \mathtt{w}'_2) \in [\![\sigma\tau_1]\!]_\mathtt{v}$ (obtained by Lemma 3 on ($\dagger\dagger$) using $m - f_1 - f_2 - c_{app}(\mathbb{C}, \mathbb{S}) < m - f_2$), we get
$(m - f_1 - f_2 - c_{app}(\mathbb{C}, \mathbb{S}), \mathtt{ee}[(\mathtt{fix\ f(x).\ ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}]) \in [\![\sigma\tau_2]\!]^{\sigma\kappa'}_\varepsilon$.
Unrolling its definition with the premise ($\star\star\star$), noting that
$\mathrm{L}(\mathtt{ee}[(\mathtt{fix\ f(x).\ ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}]) = \mathtt{e}[\mathtt{v}_2/\mathtt{x}, (\mathtt{fix\ f(x).\ e})/\mathtt{f}]$ from c) and h), and
$f_r < m - f_1 - f_2 - c_{app}(\mathbb{C}, \mathbb{S})$ since $f < m$, we get

p) $\langle T_r, \mathtt{ee}[(\mathtt{fix\ f(x).\ ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}]\rangle \curvearrowright \mathtt{w}'_\mathtt{r}, \mathtt{T}'_\mathtt{r}, \mathtt{c}'_\mathtt{r}$

q) $\mathrm{R}(\mathtt{ee}[(\mathtt{fix\ f(x).\ ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}] = \mathtt{e}'[(\mathtt{fix\ f(x).\ e}')/\mathtt{f}, \mathtt{v}'_2/\mathtt{x}]) \Downarrow \mathtt{T}'_\mathtt{r}, \mathtt{f}'_\mathtt{r}$

r) $v_r = \mathrm{L}(\mathtt{w}'_\mathtt{r}) \ \wedge\ \mathtt{V}(\mathtt{T}'_\mathtt{r}) = \mathtt{v}'_\mathtt{r} = \mathrm{R}(\mathtt{w}'_\mathtt{r})$

s) $c'_r \leq \sigma\kappa'$

t) $(m - f_1 - f_2 - f_r - c_{app}(\mathbb{C}, \mathbb{S}), \mathtt{w}'_\mathtt{r}) \in [\![\sigma\tau_2]\!]_\mathtt{v}$

1. By a), f) and k)

$$\frac{\begin{array}{c}\langle T_1, \theta^{\ulcorner} e_1 {}^{\urcorner}\rangle \curvearrowright \texttt{fix}\ f(x).\texttt{ee}, \mathtt{T}'_1, \mathtt{c}'_1 \\ \langle T_2, \theta^{\ulcorner} e_2 {}^{\urcorner}\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2 \qquad \langle T_r, \texttt{ee}[(\texttt{fix f(x).ee})/\mathtt{f}, \mathtt{w}'_2/\mathtt{x}]\rangle \curvearrowright \mathtt{w}'_\mathtt{r}, \mathtt{T}'_\mathtt{r}, \mathtt{c}'_\mathtt{r} \qquad v'_r = \mathtt{V}(T'_r)\end{array}}{\langle\langle \_, \texttt{app}(T_1, T_2, T_r)\rangle, \theta^{\ulcorner} e_1\ e_2 {}^{\urcorner}\rangle \curvearrowright \mathtt{w}'_\mathtt{r}, \langle \mathtt{v}'_\mathtt{r}, \texttt{app}(\mathtt{T}'_1, \mathtt{T}'_2, \mathtt{T}'_\mathtt{r})\rangle, \mathtt{c}'_1 + \mathtt{c}'_2 + \mathtt{c}'_\mathtt{r}}\ \textbf{r-app1}$$

2. By b), g) and l)

$$\frac{\begin{array}{c}\mathrm{R}(\theta^{\ulcorner} e_1 {}^{\urcorner}) \Downarrow T'_1, f'_1 \qquad \texttt{fix}\ f(x).e' = \mathtt{V}(T'_1) \\ \mathrm{R}(\theta^{\ulcorner} e_2 {}^{\urcorner}) \Downarrow T'_2, f'_2 \qquad v'_2 = \mathtt{V}(T'_2) \qquad e'[(\texttt{fix}\ f(x).e')/f, v'_2/x] \Downarrow T'_r, f'_r \qquad v'_r = \mathtt{V}(T'_r)\end{array}}{\mathrm{R}(\theta^{\ulcorner} e_1\ e_2 {}^{\urcorner}) \Downarrow \langle v'_r, \texttt{app}(T'_1, T'_2, T'_r)\rangle, f'_1 + f'_2 + f'_r + c_{app}(\mathbb{C}, \mathbb{S})}\ \textbf{app}$$

3. follows immediately from m)
4. By d), i), n) $c_1 + c_2 + c_r \leq \sigma(\kappa_1 + \kappa_2 + \kappa')$
5. Since $f = f_1 + f_2 + f_r + c_{app}(\mathbb{C}, \mathbb{S})$, by o), we get $(m - f, \mathtt{w}'_\mathtt{r}) \in [\![\sigma\tau_2]\!]_\mathtt{v}$, noting that $m - f = m - f_1 - f_2 - f_r - c_{app}(\mathbb{C}, \mathbb{S})$.

subcase 3: $\mu = \mathbb{C}$ and $\delta = \mathbb{C}$

By e), we have $(m - f_1, \mathtt{w}'_1) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2)^{\mathbb{C}}]\!]_\mathtt{v}$.

There are two cases.

- $(m - f_1, \texttt{new}(\texttt{fix}\ f(x).e, \texttt{fix}\ f(x).e')) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2)^{\mathbb{C}}]\!]_v$

  By unrolling the first part of the definition, we have

  $\forall k.\ (k, \texttt{fix}\ f(x).e) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2|\!)_v$ ($\diamond$) $\wedge$ $(k, \texttt{fix}\ f(x).e') \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2|\!)_v$ ($\diamond\diamond$).

  By Lemma 2 on $(m - f_2, \mathtt{w}'_2) \in [\![\sigma\tau_1]\!]_\mathtt{v}$ obtained by j), we get

  $\forall t.\ (t, \mathrm{L}(\mathtt{w}'_2)) \in (\!|\sigma\tau_1|\!)_\mathtt{v}$ ($\dagger$) $\wedge$ $(\mathtt{t}, \mathrm{R}(\mathtt{w}'_2)) \in (\!|\sigma\tau_1|\!)_\mathtt{v}$ ($\dagger\dagger$).

  Next, we instantiate ($\diamond\diamond$) with step index $\sigma\kappa' + 2$, and ($\dagger\dagger$) with step index $\sigma\kappa' + 2$ to get the cost f'_r as follows:

  Unroll $(\sigma\kappa' + 2, \texttt{fix}\ f(x).e') \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2|\!)_v$ with $(\sigma\kappa' + 1, \mathrm{R}(\mathtt{w}'_2)) \in (\!|\sigma\tau_1|\!)_\mathtt{v}$ since $\sigma\kappa' + 1 < \sigma\kappa' + 2$, to get $(\sigma\kappa' + 1, e'[\texttt{fix}\ f(x).e'/f, \mathrm{R}(\mathtt{w}'_2)/\mathtt{x}]) \in (\!|\sigma\tau_2|\!)^{\sigma\kappa'}_\varepsilon$.

  Next, we unroll the definition of $(\!|\cdot|\!)^{\cdot}_\varepsilon$ using $\sigma\kappa' < \sigma\kappa' + 1$ to obtain:

  $e'[\texttt{fix}\ f(x).e'/f, \mathrm{R}(\mathtt{w}'_2)/\mathtt{x}] \Downarrow \mathtt{T}'_\mathtt{r}, \mathtt{f}'_\mathtt{r}$ ($\spadesuit$) and $v'_r = \mathtt{V}(T'_r)$.

  The statement ($\diamond\diamond$) will be re-instantiated again to get the full proof.

  We proceed to the remaining statements to show:

  1. By a), f) and ($\spadesuit$)

$$\frac{\begin{array}{c}\langle T_1, \theta^{\ulcorner} e_1 {}^{\urcorner}\rangle \curvearrowright \texttt{new}(\_, \texttt{fix}\ f(x).e'), T'_1, c'_1 \\ \langle T_2, \theta^{\ulcorner} e_2 {}^{\urcorner}\rangle \curvearrowright \mathtt{w}'_2, \mathtt{T}'_2, \mathtt{c}'_2 \qquad e'[\texttt{fix}\ f(x).e'/f, \mathrm{R}(\mathtt{w}'_2)/\mathtt{x}] \Downarrow \mathtt{T}'_\mathtt{r}, \mathtt{f}'_\mathtt{r} \qquad v'_r = \mathtt{V}(T'_r)\end{array}}{\begin{array}{c}\langle\langle v_r, \texttt{app}(T_1, T_2, T_r)\rangle, \theta^{\ulcorner} e_1\ e_2 {}^{\urcorner}\rangle \curvearrowright \\ \texttt{new}(v_r, v'_r), \langle v'_r, \texttt{app}(T'_1, T'_2, T'_r)\rangle, c'_1 + c'_2 + f'_r + c_{app}(\mathbb{S}, \mathbb{C})\end{array}}\ \textbf{r-app2}$$

  2. By b), g) and ($\spadesuit$)

$$\frac{\mathrm{R}(\theta^{\ulcorner} e_1 {}^{\urcorner}) \Downarrow T_1', f_1' \qquad \texttt{fix}\ f(x).\, e' = \mathtt{V}(T_1') \qquad \mathrm{R}(\theta^{\ulcorner} e_2 {}^{\urcorner}) \Downarrow T_2', f_2' \qquad v_2' = \mathtt{V}(T_2') \qquad e'[(\texttt{fix}\ f(x).\, e')/f, v_2'/x] \Downarrow T_r', f_r' \qquad v_r' = \mathtt{V}(T_r')}{\mathrm{R}(\theta^{\ulcorner} e_1\ e_2 {}^{\urcorner}) \Downarrow \langle v_r', \mathtt{app}(T_1', T_2', T_r')\rangle, f_1' + f_2' + f_r' + c_{app}(\mathbb{C}, \mathbb{S})}\ \textbf{app}$$

3. follows immediately: $v_r = \mathrm{L}(\mathtt{new}(v_r, v_r'))$ and $v_r' = \mathrm{R}(\mathtt{new}(v_r, v_r'))$

The statements 4. and 5. will be shown below.

We first show statement 5:

TS:$(m - j, \mathtt{new}(v_r, v_r')) \in [\![\sigma\tau_2]\!]_v$.

By using the premise $\models \mathbb{C} \trianglelefteq \tau_2$ of the typing judgment, $\tau_2 = (A)^{\mathbb{C}}$

STS: $\forall k.\ (k, v_r) \in (\!|\sigma\tau_2|\!)_v \ \wedge\ (k, v_r') \in (\!|\sigma\tau_2|\!)_v$.

Pick some $\sigma\kappa'$, then

STS1: $(\sigma\kappa', v_r) \in (\!|\sigma\tau_2|\!)_v(\spadesuit\spadesuit)$.

STS2: $(\sigma\kappa', v_r') \in (\!|\sigma\tau_2|\!)_v(\spadesuit\spadesuit\spadesuit)$.

We first show the first one.

Next, we instantiate ($\diamond$) with $\sigma\kappa' + f_r + 2$ and ($\dagger$) with $\sigma\kappa' + f_r + 1$.

Then, we unroll $(\sigma\kappa' + f_r + 2, \texttt{fix}\ f(x).\, e) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2|\!)_v$ with

$(\sigma\kappa' + f_r + 1, \mathrm{L}(\mathtt{w}_2')) \in (\!|\sigma\tau_1|\!)_{\mathtt{v}}$ since $\sigma\kappa' + f_r + 1 < \sigma\kappa' + f_r + 2$ and we get:

$(\sigma\kappa' + f_r + 1, e[\texttt{fix}\ f(x).\, e/f, \mathrm{L}(\mathtt{w}_2')/\mathtt{x}]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma\kappa'}$.

Unrolling the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$ using $\sigma\kappa' < \sigma\kappa' + f_r + 1$, we obtain:

u) $e[\texttt{fix}\ f(x).\, e/f, \mathrm{L}(\mathtt{w}_2)/\mathtt{x}] \Downarrow \mathtt{T_r}, \mathtt{f_r}$ and $v_r = \mathtt{V}(T_r)$

v) $(\sigma\kappa' + 1, v_r) \in (\!|\sigma\tau_2|\!)_v$

w) $f_r < \sigma\kappa'$

Next, we instantiate ($\diamond\diamond$) with $\sigma\kappa' + f_r' + 2$ and ($\dagger\dagger$) with $\sigma\kappa' + f_r' + 1$. Note that at this point, we know what f_r' is.

Then, we unroll $(\sigma\kappa' + f_r' + 2, \texttt{fix}\ f(x).\, e') \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2|\!)_v$ with

$(\sigma\kappa' + f_r' + 1, \mathrm{R}(\mathtt{w}_2')) \in (\!|\sigma\tau_1|\!)_{\mathtt{v}}$ since $\sigma\kappa' + f_r' + 1 < \sigma\kappa' + f_r' + 2$, we get:

$(\sigma\kappa' + f_r' + 1, e'[\texttt{fix}\ f(x).\, e'/f, \mathrm{R}(\mathtt{w}_2')/\mathtt{x}]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma\kappa'}$.

Unrolling the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$ with $\sigma\kappa' < \sigma\kappa' + f_r' + 1$, we obtain:

x) $e'[\texttt{fix}\ f(x).\, e'/f, \mathrm{R}(\mathtt{w}_2)/\mathtt{x}] \Downarrow \mathtt{T_r'}, \mathtt{f_r'}$ and $v_r' = \mathtt{V}(T_r')$

y) $(\sigma\kappa' + 1, v_r') \in (\!|\sigma\tau_2|\!)_v$

z) $f_r' < \sigma\kappa'$

Now, we can conclude the statement 5. by instantiating downward closure (Lemma 3) on q), t) and $\sigma\kappa' \leq k + \sigma\kappa'$ to obtain ($\spadesuit\spadesuit$) and ($\spadesuit\spadesuit\spadesuit$).

The statement 4. follows by d), i) and z), $c_1 + c_2 + f_r + c_{app}(\mathbb{S}, \mathbb{C}) \leq \sigma(\kappa_1 + \kappa_2 + \kappa') + c_{app}(\mathbb{S}, \mathbb{C})$.

- $(m - f_1, \mathtt{w}_1') \in [\![\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2]\!]_{\mathtt{v}}$.

Proof of this subcase is same as the proof of the subcase with $\mu = \mathbb{S}$ and $\delta = \mathbb{C}$.

Case $\dfrac{t :: S, \Delta; \Phi; \Gamma \vdash_{\delta} e : \tau \mid \kappa}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \Lambda.\, e : (\forall t \overset{\delta(\kappa)}{::} S.\ \tau)^{\mathbb{S}} \mid 0}\ \forall\mathbf{I}$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner}\Lambda.\, e^{\urcorner}) \in [\![(\forall t \overset{\delta(\sigma\kappa)}{::} S.\ \sigma\tau)^{\mathbb{S}}]\!]^{0}_{\varepsilon}$.

STS: $(m, \theta^{\ulcorner}\Lambda.\, e^{\urcorner}) \in [\![\forall t \overset{\delta(\sigma\kappa)}{::} S.\ \sigma\tau]\!]_{v}$ by Lemma 5.

There are two cases:

subcase 1: $\delta = \mathbb{S}$

Assume that $\vdash I :: S$.

STS: $(m, \theta^{\ulcorner}e^{\urcorner}) \in [\![\sigma\tau\{I/t\}]\!]^{\sigma\kappa\{I/t\}}_{\varepsilon}$

This follows from the IH 1 instantiated with the substitution $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$ where $\sigma[t \mapsto I]\delta = \mathbb{S}$ since $t \notin FV(\delta)$ and $\delta = \mathbb{S}$. Note that $\models \sigma[t \mapsto I]\Phi$ is same as $\models \sigma\Phi$ since $t \notin FV(\Phi; \Gamma)$.

subcase 2: $\delta = \mathbb{C}$

- STS1: $\forall k.\ (k, \mathrm{L}(\theta^{\ulcorner}\Lambda.\, e^{\urcorner})) \in (\!|(\forall t \overset{\mathbb{C}(\sigma\kappa)}{::} S.\ \sigma\tau)^{\mathbb{S}}|\!)_{v} \wedge (k, \mathrm{R}(\theta^{\ulcorner}\Lambda.\, e^{\urcorner})) \in (\!|(\forall t \overset{\mathbb{C}(\sigma\kappa)}{::} S.\ \sigma\tau)^{\mathbb{S}}|\!)_{v}$

  Pick some k and assume that $\vdash I :: S$, then

  STS1: $(k, \mathrm{L}(\theta^{\ulcorner}e^{\urcorner})) \in (\!|\sigma\tau[I/t]|\!)^{\sigma\kappa[I/t]}_{\varepsilon}$ ($\diamond$)

  STS2: $(k, \mathrm{R}(\theta^{\ulcorner}e^{\urcorner})) \in (\!|\sigma\tau[I/t]|\!)^{\sigma\kappa[I/t]}_{\varepsilon}$ ($\diamond\diamond$).

  We first show the first one.

  By IH 2 on the premise of the typing rule, instantiated with

  - $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$
  - $(k, \mathrm{L}(\theta)) \in \mathcal{G}(\!|\Gamma|\!)$ obtained by instantiating Lemma 2 on $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ with k. We get $(k, \mathrm{L}(\theta)(e)) \in (\!|\sigma[t \mapsto I]\tau|\!)^{\sigma[t \mapsto I]\kappa}_{\varepsilon}$ which is same as ($\diamond$) since $\mathrm{L}(^{\ulcorner}e^{\urcorner}) = e$

  By IH 2 on the premise of the typing rule, instantiated with

  - $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$
  - $(k, \mathrm{R}(\theta)) \in \mathcal{G}(\!|\Gamma|\!)$ obtained by instantiating Lemma 2 on $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ with k. We get $(k, \mathrm{R}(\theta)(e)) \in (\!|\sigma[t \mapsto I]\tau|\!)^{\sigma[t \mapsto I]\kappa}_{\varepsilon}$ which is same as ($\diamond\diamond$) since $\mathrm{R}(^{\ulcorner}e^{\urcorner}) = e$

- STS2: $\forall I.\ \vdash I :: S \Rightarrow (m, \theta^{\ulcorner}\Lambda.\, e^{\urcorner}) \in [\![\tau[I/t]]\!]^{\kappa[I/t]}_{\varepsilon}$

  Assume that $\vdash I :: S$, then STS: $(m, \mathtt{ee}) \in [\![\tau[\mathtt{I}/\mathtt{t}]]\!]^{\kappa[\mathtt{I}/\mathtt{t}]}_{\varepsilon}$.

  This follows from the IH 3 instantiated with the substitution $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$.

  .

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : (\forall t \overset{\delta(\kappa')}{::} S.\ \tau)^{\mu} \mid \kappa_e \qquad \Delta \vdash I :: S \qquad \Delta; \Phi \models \mu \trianglelefteq \tau\{I/t\} \qquad \Delta; \Phi \models (\epsilon \sqcup \mu) \leq \delta \qquad \kappa = \kappa_e + \kappa'\{I/t\} + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{iApp}(\epsilon, \mu)\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e[] : \tau\{I/t\} \mid \kappa}\ \forall\mathbf{E}$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner} e[]^{\urcorner}) \in [\![\sigma\tau\{\sigma I/t\}]\!]^{\sigma\kappa}_{\varepsilon}$

Following the definition of $[\![\cdot]\!]^{:}_{\varepsilon}$, assume that

$$\frac{\mathrm{L}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T, f_e \ (\star) \qquad \Lambda.\, e' = \mathtt{V}(T) \qquad e' \Downarrow T_r, f_r \ (\star\star) \qquad v_r = \mathtt{V}(T_r)}{\mathrm{L}(\theta^{\ulcorner} e[]^{\urcorner}) \Downarrow \langle v_r, \mathtt{iApp}(T, T_r)\rangle, f_e + f_r + c_{iApp}(\mathbb{C}, \_)}\ \textbf{iApp}$$

where $f = f_e + f_r + c_{iApp}(\mathbb{C}, \_) < m$

By IH on e, $(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![(\forall t \overset{\delta(\sigma\kappa')}{::} S.\ \sigma\tau)^{\mu}]\!]^{\sigma\kappa_e}_{\varepsilon}$.

Unrolling this, using the premise marked $\star$ with the definition $f_e < f < m$, we get

a) $\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$

b) $\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f'$

c) $\Lambda.\, e' = \mathrm{L}(\mathtt{w}') \ \wedge \ \mathtt{V}(\mathtt{T}') = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$

d) $c' \leq \sigma\kappa_e$

e) $(m - f_e, \mathtt{w}') \in [\![(\forall \mathtt{t} \overset{\delta(\sigma\kappa')}{::} \mathtt{S}.\ \sigma\tau)^{\mu}]\!]_{\mathtt{v}}$

By Lemma 1, $\vdash \sigma I :: S$ (†).

There are 3 cases for e).

subcase 1: $\delta = \mathbb{S}$ and $\mu = \mathbb{S}$

Then we have, $(m - f_e, \Lambda.\mathfrak{ee}) \in [\![(\forall \mathtt{t} \overset{\mathbb{S}(\sigma\kappa')}{::} \mathtt{S}.\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$ (⋄) for some $\mathfrak{ee}$.

Unrolling (⋄) with $\vdash \sigma I :: S$ (†), we get $(m - f_e, \mathfrak{ee}) \in [\![\sigma\tau\{\sigma\mathtt{I}/\mathtt{t}\}]\!]^{\sigma\kappa'\{\sigma\mathtt{I}/\mathtt{t}\}}_{\varepsilon}$.

Unrolling this with the premise marked $(\star\star)$ and defining $f_r < m - f_e$, we get

f) $\langle T_r, \mathfrak{ee}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}}$

g) $\mathrm{R}(\mathfrak{ee}) \Downarrow \mathtt{T}'_{\mathtt{r}}, \mathtt{f}'_{\mathtt{r}}$

h) $v_r = \mathrm{L}(\mathtt{w}'_{\mathtt{r}}) \ \wedge \ \mathtt{V}(\mathtt{T}'_{\mathtt{r}}) = \mathtt{v}'_{\mathtt{r}} = \mathrm{R}(\mathtt{w}'_{\mathtt{r}})$

i) $c'_r \leq \sigma\kappa'\{\sigma I/t\}$

j) $(m - f_e - f_r, \mathtt{w}_{\mathtt{r}}) \in [\![\sigma\tau\{\sigma\mathtt{I}/\mathtt{t}\}]\!]_{\mathtt{v}}$

1. By a) and f)
$$\frac{\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \Lambda.\mathfrak{ee}, \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \mathfrak{ee}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}} \qquad \mathtt{V}(T'_r) = v'_r}{\langle\langle \_, \mathtt{iApp}(T, T_r)\rangle, \theta^{\ulcorner} \mathfrak{ee}[]^{\urcorner}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \langle \mathtt{v}'_{\mathtt{r}}, \mathtt{iApp}(\mathtt{T}', \mathtt{T}'_{\mathtt{r}})\rangle, \mathtt{c}' + \mathtt{c}'_{\mathtt{r}}}\ \textbf{r-App1}$$
2. By b) and g)
$$\frac{\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow \Lambda.\, e'', T' f' \qquad e'' \Downarrow T'_r, f'_r \qquad \mathtt{V}(T'_r) = v'_r}{\mathrm{R}(\theta^{\ulcorner} e[]^{\urcorner}) \Downarrow \langle v'_r, \mathtt{iApp}(T', T'_r)\rangle, f' + f'_r + c_{iApp}(\mathbb{C}, \_)}\ \textbf{iApp}$$
3. follows immediately from h)
4. By d) and i), $c' + c'_r \leq \sigma\kappa + \sigma\kappa'\{\sigma I/t\}$)
5. By Lemma 3 using j)

subcase 2: $\delta = \mathbb{C}$ and $\mu = \mathbb{S}$

The proof of this case is very similar to the above case, we will show it for clarity.

Then we have, $(m - f_e, \Lambda.\mathtt{ee}) \in [\![(\forall \mathtt{t} \overset{\mathbb{C}(\sigma\kappa')}{::} \mathtt{S}.\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$ ($\diamond$) for some $\mathtt{ee}$.

Unrolling **second part of** ($\diamond$) with $\vdash \sigma I :: S$ ($\dagger$), we get

$(m - f_e, \mathtt{ee}) \in [\![\sigma\tau\{\sigma\mathtt{I}/\mathtt{t}\}]\!]_{\varepsilon}^{\sigma\kappa'\{\sigma\mathtt{I}/\mathtt{t}\}}$.

Unrolling this with the premise marked ($\star\star$) and defining $f_r < m - f_e$, we get

k) $\langle T_r, \mathtt{ee}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}}$

l) $\mathrm{R}(\mathtt{ee}) \Downarrow \mathtt{T}'_{\mathtt{r}}, \mathtt{f}'_{\mathtt{r}}$

m) $v_r = \mathrm{L}(\mathtt{w}'_{\mathtt{r}}) \ \wedge\ \mathtt{V}(\mathtt{T}'_{\mathtt{r}}) = \mathtt{v}'_{\mathtt{r}} = \mathrm{R}(\mathtt{w}'_{\mathtt{r}})$

n) $c'_r \leq \sigma\kappa'\{\sigma I/t\}$

o) $(m - f_e - f_r, \mathtt{w}_{\mathtt{r}}) \in\ [\![\sigma\tau\{\sigma\mathtt{I}/\mathtt{t}\}]\!]_{\mathtt{v}}$

1. By a) and f)
$$\frac{\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \Lambda.\mathtt{ee}, \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \mathtt{ee}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}} \qquad \mathtt{V}(T'_r) = v'_r}{\langle\langle \_, \mathtt{iApp}(T, T_r)\rangle, \theta^{\ulcorner} \mathtt{ee}[]^{\urcorner}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \langle \mathtt{v}'_{\mathtt{r}}, \mathtt{iApp}(\mathtt{T}', \mathtt{T}'_{\mathtt{r}})\rangle, \mathtt{c}' + \mathtt{c}'_{\mathtt{r}}}\ \textbf{r-App1}$$
2. By b) and g)
$$\frac{\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow \Lambda.\, e'', T' f' \qquad e'' \Downarrow T'_r, f'_r \qquad \mathtt{V}(T'_r) = v'_r}{\mathrm{R}(\theta^{\ulcorner} e[]^{\urcorner}) \Downarrow \langle v'_r, \mathtt{iApp}(T', T'_r)\rangle, f' + f'_r + c_{iApp}(\mathbb{C}, \_)}\ \textbf{iApp}$$
3. follows immediately from h)
4. By d) and i), $c' + c'_r \leq \sigma\kappa + \sigma\kappa'\{\sigma I/t\}$)
5. By Lemma 3 using j)

subcase 3: $\delta = \mathbb{C}$ and $\mu = \mathbb{C}$

Then by e) we have, $(m - f_e, \mathtt{w}') \in [\![(\forall \mathtt{t} \overset{\mathbb{C}(\sigma\kappa')}{::} \mathtt{S}.\ \sigma\tau)^{\mathbb{C}}]\!]_{\mathtt{v}}$.

There are two cases.

- $(m - f, \mathtt{new}(\Lambda.\, e, \Lambda.\, e'')) \in [\![(\forall t \overset{\mathbb{C}(\sigma\kappa')}{::} S.\ \sigma\tau)^{\mathbb{C}}]\!]_v$.

  By unrolling the definition, we have

  $\forall k.\ (k, \Lambda.\, e) \in\ (\!|\forall t \overset{\mathbb{C}(\sigma\kappa')}{::} S.\ \sigma\tau|\!)_v$ ($\diamond$) $\wedge$ $(k, \Lambda.\, e'') \in\ (\!|\forall t \overset{\mathbb{C}(\sigma\kappa')}{::} S.\ \sigma\tau|\!)_v$ ($\diamond\diamond$).

  We first unroll ($\diamond\diamond$).

  Assume that $k = \sigma\kappa' + 1$. Then, we have $(\sigma\kappa' + 1, \Lambda.\, e'') \in\ (\!|\forall t \overset{\mathbb{C}(\sigma\kappa')}{::} S.\ \sigma\tau|\!)_v$.

  Unrolling this with $\vdash \sigma I :: S$ ($\dagger$), we get $(\sigma\kappa' + 1, e'') \in\ (\!|\sigma\tau\{\sigma I/t\}|\!)_{\varepsilon}^{\sigma\kappa'\{\sigma I/t\}}$ ($\spadesuit$).

  By unrolling the definition of ($\spadesuit$) using $\sigma\kappa' < \sigma\kappa' + 1$, we get

  $e'' \Downarrow T'_r, f'_r$ ($\dagger\dagger$) and $v'_r = \mathtt{V}(T'_r)$.

  The statement ($\diamond\diamond$) will be unrolled again to get the full proof.

  Then, we can show:

  1. By a) and ($\dagger\dagger$)
  $$\frac{\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{new}(\_, \Lambda.\, e''), T', c' \qquad e'' \Downarrow T'_r, f'_r \qquad v'_r = \mathtt{V}(T'_r)}{\langle\langle v_r, \mathtt{iApp}(T, T_r)\rangle, \theta^{\ulcorner} \mathtt{ee}[]^{\urcorner}\rangle \curvearrowright \mathtt{new}(\mathtt{v_r}, \mathtt{v}'_{\mathtt{r}}), \langle \mathtt{v}'_{\mathtt{r}}, \mathtt{iApp}(\mathtt{T}', \mathtt{T}'_{\mathtt{r}})\rangle, \mathtt{c}' + \mathtt{f}'_{\mathtt{r}} + \mathtt{c}_{\mathtt{iApp}}(\mathbb{S}, \mathbb{C})}\ \textbf{r-App2}$$
  2. By b) and ($\dagger\dagger$)

$$\frac{\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f' \qquad \Lambda.\, e'' = \mathtt{V}(T') \qquad e'' \Downarrow T'_r, f'_r \qquad v'_r = \mathtt{V}(T'_r)}{\mathrm{R}(\theta^{\ulcorner} e[]^{\urcorner}) \Downarrow \langle v'_r, \mathtt{iApp}(T', T'_r)\rangle, f' + f'_r + c_{iApp}(\mathbb{C}, \_)}\ \textbf{App}$$

3. follows immediately: $v_r = \mathrm{L}(\mathtt{new}(v_r, v'_r))$ and $v'_r = \mathrm{R}(\mathtt{new}(v_r, v'_r))$

The statements 4. and 5. will be shown below.

We first show statement 5.

TS: $(m - f_e - f_r - c_{iApp}(\mathbb{C}, \_), \mathtt{new}(v_r, v'_r)) \in [\![\sigma\tau\{\sigma I/t\}]\!]_v$ where by the premise $\models \mathbb{C} \trianglelefteq \tau$ of the typing judgment, $\sigma\tau\{\sigma I/t\} = (A)^{\mathbb{C}}$ for some type A.

Then,

TS: $\forall k.\ (k, v'_r) \in (\!|A|\!)_v \ \wedge\ (k, v'_r) \in (\!|A|\!)_v$.

Pick some k, then STS: $(k, v'_r) \in (\!|A|\!)_v$(†††).

Next, we instantiate ($\diamond$) with $\sigma\kappa' + f'_r + 1$ and get:

$(\sigma\kappa' + f'_r + 1, \Lambda.\, e'') \in (\!|\forall t \overset{\mathbb{C}(\sigma\kappa')}{::} S.\ \sigma\tau|\!)_v$.

Unrolling with (†), we get : $(\sigma\kappa' + f'_r + 1, e'') \in (\!|\sigma\tau\{\sigma I/t\}|\!)^{\sigma\kappa'}_{\varepsilon}$.

Unrolling the definition of $(\!|\cdot|\!)^{\cdot}_{\varepsilon}$ with $\sigma\kappa' < \sigma\kappa' + f'_r + 1$, we obtain:

p) $e'' \Downarrow T'_r, f'_r$

q) $(\sigma\kappa' + 1, v'_r) \in (\!|A|\!)_v$

r) $f'_r < \sigma\kappa'$

Now, we can conclude the statement 5. by instantiating downward closure (Lemma 3) on l) and $\sigma\kappa' \leq k + \sigma\kappa'$ to obtain (†††).

The statement 4. follows by d) and m), $c' + f'_r + c_{iApp}(\mathbb{S}, \mathbb{C}) \leq \sigma(\kappa_e + \kappa') + c_{iApp}(\mathbb{S}, \mathbb{C})$.

- $(m - f, \Lambda.\mathtt{ee}) \in [\![\forall \mathtt{t} \overset{\mathbb{C}(\sigma\kappa')}{::} \mathtt{S}.\ \sigma\tau]\!]_{\mathtt{v}}$

  Proof of this subcase is similar to the proof of the subcase with $\delta = \mathbb{C}$ and $\mu = \mathbb{S}$.

$$\textbf{Case}\ \frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : (\tau_1 + \tau_2)^{\mu} \mid \kappa_e \qquad \Delta; \Phi; \Gamma, x : \tau_1 \vdash_{\epsilon \sqcup \mu} e_1 : \tau \mid \kappa' \\ \Delta; \Phi; \Gamma, y : \tau_2 \vdash_{\epsilon \sqcup \mu} e_2 : \tau \mid \kappa' \qquad \models \mu \trianglelefteq \tau \qquad \kappa = \kappa_e + \kappa' + ((\mu \doteq \mathbb{C})\ ?\ c_{case}(\mathbb{S}, \mu) :\ 0)\end{array}}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \mathtt{case}(e, x.e_1, y.e_2) : \tau \mid \kappa}\ \textbf{case}$$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner} \mathtt{case}(e, x.e_1, y.e_2)^{\urcorner}) \in [\![\sigma\tau]\!]^{\sigma\kappa}_{\varepsilon}$

Following the definition of $[\![\cdot]\!]^{\cdot}_{\varepsilon}$, there are two cases. We will only show the case where guard evaluates to $\mathtt{inl}\ v$, the other case is symmetric.

Assume that:

$$\frac{\mathrm{L}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T, f_e(\star) \qquad \mathtt{inl}\ v = \mathtt{V}(T) \qquad \mathrm{L}(\theta^{\ulcorner} e_1{}^{\urcorner})[v/x] \Downarrow T_r, f_r(\star\star) \qquad v_r = \mathtt{V}(T_r)}{\mathrm{L}(\theta^{\ulcorner} \mathtt{case}(e, x.e_1, y.e_2)^{\urcorner}) \Downarrow \langle v_r, \mathtt{case}_{\mathtt{inl}}(T, T_r)\rangle, f_e + f_r + c_{case}(\mathbb{C}, \_)}\ \textbf{r-case-inl}$$

where $f = f_e + f_r + c_{case}(\mathbb{C}, \_) < m$

By IH on e, $(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![(\sigma\tau_1 + \sigma\tau_2)^{\mu}]\!]^{\sigma\kappa_e}_{\varepsilon}$.

Unrolling this, using the premise marked $\star$ with the definition $f_e < f < m$, we get

a) $\langle T, \theta^{\ulcorner} e^{\urcorner} \rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$

b) $\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f'$

c) $\mathtt{inl}\ v = \mathrm{L}(\mathtt{w}') \ \wedge\ \mathtt{V}(\mathtt{T}') = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$

d) $c' \leq \sigma\kappa_e$

e) $(m - f_e, \mathtt{w}') \in [\![(\sigma\tau_1 + \sigma\tau_2)^{\mu}]\!]_{\mathtt{v}}$

There are two cases for e).

subcase 1: $\mu = \mathbb{S}$

Then, we have $(m - f_e, \mathtt{inl}\ \mathtt{w}) \in [\![\sigma\tau_1 + \sigma\tau_2]\!]_{\mathtt{v}}$ and $v' = \mathtt{inl}\ v''$ for some v''.
By unrolling the definition, we get $(m - f_e, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$ ($\diamond$). Next, we instantiate the IH 1 on the premise $\Delta; \Phi; \Gamma, x : \tau_1 \vdash_{\epsilon \sqcup \mu} e_1 : \tau \mid \kappa'$ using $\epsilon \sqcup \mu = \mathbb{S}$ and $(m - f_e, \theta[x \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma\Gamma, \mathtt{x} : \sigma\tau_1]\!]$, which holds because

* $(m - f_e, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $m - f_e \leq m$,
* by e)

We get, $(m - f_e, \theta[x \mapsto \mathtt{w}]^{\ulcorner} \mathtt{e_1}^{\urcorner}) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa'}$ ($\diamond\diamond$).
Then, by unrolling ($\diamond\diamond$) with ($\star\star$) where $\mathrm{L}(\theta[x \mapsto \mathtt{w}]^{\ulcorner} \mathtt{e_1}^{\urcorner}) = \mathrm{L}(\theta^{\ulcorner} \mathtt{e_1}^{\urcorner}[\mathtt{v}/\mathtt{x}])$ (by c), $v = \mathrm{L}(\mathtt{w})$) and $f_r < m - f_e$, we get

f) $\langle T_1, \theta[x \mapsto \mathtt{w}]^{\ulcorner} \mathtt{e}^{\urcorner} \rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}}$

g) $\mathrm{R}(\theta[x \mapsto \mathtt{w}]^{\ulcorner} \mathtt{e}^{\urcorner}) = \mathrm{R}(\theta^{\ulcorner} \mathtt{e_1}^{\urcorner}[\mathtt{v}''/\mathtt{x}]) \Downarrow \mathtt{T}'_{\mathtt{r}}, \mathtt{f}'_{\mathtt{r}}$

h) $v_r = \mathrm{L}(\mathtt{w}') \ \wedge\ \mathtt{V}(\mathtt{T}'_{\mathtt{r}}) = \mathtt{v}'_{\mathtt{r}} = \mathrm{R}(\mathtt{w}')$

i) $c' \leq \sigma\kappa'$

j) $(m - f_e - f_r, \mathtt{w}'_{\mathtt{r}}) \in [\![\sigma\tau]\!]_{\mathtt{v}}$

Then, we conclude this subcase by showing

1. By a), f) and h)
$$\frac{\langle T_1, \theta^{\ulcorner} e^{\urcorner} \rangle \curvearrowright \mathtt{inl}\ \mathtt{w}, \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \theta^{\ulcorner} e_1{}^{\urcorner}[\mathtt{w}/\mathtt{x}] \rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}}}{\langle \langle \_, \mathtt{case}_{\mathtt{inl}}(T, T_r) \rangle, \theta^{\ulcorner} \mathtt{case}(e, x.e_1, y.e_2)^{\urcorner} \rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \langle \mathtt{v}'_{\mathtt{r}}, \mathtt{case}_{\mathtt{inl}}(\mathtt{T}', \mathtt{T}'_{\mathtt{r}}) \rangle, \mathtt{c}' + \mathtt{c}'_{\mathtt{r}}}\ \textbf{r-case-inl1}$$
2. By b), g) and h)
$$\frac{\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f' \qquad \mathtt{inl}\ v'' = \mathtt{V}(T') \qquad \mathrm{R}(\theta^{\ulcorner} e_1{}^{\urcorner})[v''/x] \Downarrow T'_r, f'_r \qquad v'_r = \mathtt{V}(T'_r)}{\mathrm{R}(\theta^{\ulcorner} \mathtt{case}(e, x.e_1, y.e_2)^{\urcorner}) \Downarrow \langle v'_r, \mathtt{case}_{\mathtt{inl}}(T', T'_r) \rangle, f' + f'_r + c_{case}(\mathbb{C}, \_)}\ \textbf{r-case-inl}$$
3. follows immediately from h)
4. By d) and i) and noting that $\mu = \mathbb{S}$, $c' + c'_r \leq \sigma(\kappa_e + \kappa')$
5. Since $j = f + f_r + c_{case}(\mathbb{C}, \_)$, we get $(m - f, \mathtt{w}'_{\mathtt{r}}) \in [\![\sigma\tau]\!]_{\mathtt{v}}$ by downward closure (Lemma 3) on j) and $m - f_e - f_r - c_{case}(\mathbb{C}, \_) \leq m - f_e - f_r$.

subcase 2: $\mu = \mathbb{C}$

Then, we have $(m - f_e, \mathtt{w}') \in [\![(\sigma\tau_1 + \sigma\tau_2)^{\mathbb{C}}]\!]_{\mathtt{v}}$

There are three cases.

subsubcase 1: $\mathtt{w}' = \mathtt{new}(\mathtt{inl}\ \mathtt{v}, \mathtt{inr}\ \mathtt{v}'')$ s.t. $\forall k.\ (k, \mathtt{inr}\ v) \in \lfloor\!\lfloor \sigma\tau_1 + \sigma\tau_2 \rfloor\!\rfloor_v$ and $(k, \mathtt{inr}\ v'') \in \lfloor\!\lfloor \sigma\tau_1 + \sigma\tau_2 \rfloor\!\rfloor_v$.

By unrolling their definition, we have $\forall k.\ (k, v) \in \lfloor\!\lfloor \sigma\tau_2 \rfloor\!\rfloor_v$ ($\diamond$) and $(k, v'') \in \lfloor\!\lfloor \sigma\tau_2 \rfloor\!\rfloor_v$ ($\spadesuit$).

First we instantiate the IH 2 on the premise $\Delta; \Phi; \Gamma, y : \tau_2 \vdash_{\epsilon \sqcup \mu} e_2 : \tau \mid \kappa'$ ($\diamond\diamond$) using $\epsilon \sqcup \mu = \mathbb{C}$ and $(\sigma\kappa' + 1, \mathrm{R}(\theta)[y \mapsto v'']) \in \mathcal{G}\lfloor\!\lfloor \sigma\Gamma, y : \sigma\tau_2 \rfloor\!\rfloor$, which holds because

* $(\sigma\kappa' + 1, \mathrm{R}(\theta)) \in \mathcal{G}\lfloor\!\lfloor \Gamma \rfloor\!\rfloor$ by instantiating $\forall k.(k, \mathrm{R}(\theta)) \in \mathcal{G}\lfloor\!\lfloor \Gamma \rfloor\!\rfloor$ (obtained by Lemma 2 on $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$) with $\sigma\kappa' + 1$.
* $(\sigma\kappa' + 1, v'') \in \lfloor\!\lfloor \sigma\tau_2 \rfloor\!\rfloor_v$ obtained by instantiating ($\diamond$) with $\sigma\kappa' + 1$

Then, we get $(\sigma\kappa' + 1, \mathrm{R}(\theta)[y \mapsto v'']e_2) \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_\varepsilon^{\sigma\kappa'}$.

By unrolling the definition of $\lfloor\!\lfloor \cdot \rfloor\!\rfloor_\varepsilon^{:}$ using $\sigma\kappa' < \sigma\kappa' + 1$, we get $\mathrm{R}(\theta)[y \mapsto v'']e_2 \Downarrow T_r', f_r'$ and $v_r' = \mathtt{V}(T_r')$ ($\dagger\dagger$).

The statement ($\diamond\diamond$) will be re-instantiated with IH2 again to get the full proof.

We proceed to the remaining statements to show:

1. By a) and ($\dagger\dagger$),
$$\frac{\langle T, \theta^{\ulcorner} e^{\urcorner} \rangle \curvearrowright \mathtt{new}(\mathtt{inl}\ v, \mathtt{inr}\ v''), T', c' \qquad \mathrm{R}(\theta^{\ulcorner} e_2^{\urcorner})[v''/y] \Downarrow T_r', f_r' \qquad v_r' = \mathtt{V}(T_r')}{\begin{array}{c}\langle \langle v_r, \mathtt{case}_{\mathtt{inl}}(T, T_r) \rangle, \theta^{\ulcorner} \mathtt{case}(e, x.e_1, y.e_2)^{\urcorner} \rangle \curvearrowright \\ \mathtt{new}(v_r, v_r'), \langle v_r', \mathtt{case}_{\mathtt{inr}}(T', T_r') \rangle, c' + f_r' + c_{case}(\mathbb{S}, \mathbb{C})\end{array}}\ \textbf{r-case-inr2}$$
2. By b) and ($\dagger$)
$$\frac{\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow \mathtt{inr}\ v'', T' f' \qquad \mathrm{R}(\theta^{\ulcorner} e_2^{\urcorner})[v''/y] \Downarrow v_r', T_r' f_r'}{\mathrm{R}(\theta^{\ulcorner} \mathtt{case}(e, x.e_1, y.e_2)^{\urcorner}) \Downarrow \langle v_r', \mathtt{case}_{\mathtt{inl}}(T', T_r') \rangle, f' + f_r' + c_{case}(\mathbb{C}, \_)}\ \textbf{r-inr}$$
3. follows immediately: $v_r = \mathrm{L}(\mathtt{new}(v_r, v_r'))$ and $v_r' = \mathrm{R}(\mathtt{new}(v_r, v_r'))$

The statements 4. and 5. will be shown below.

We first show statement 5.

TS: $(m - f_e - f_r - c_{case}(\mathbb{S}, \mathbb{C}), \mathtt{new}(v_r, v_r')) \in [\![\sigma\tau]\!]_v$.

By using the premise $\models \mathbb{C} \trianglelefteq \tau$ of the typing judgment, we know that $\tau = (A)^{\mathbb{C}}$,

STS: $\forall k.\ (k, v_r) \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_v$ and $(k, v_r') \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_v$.

Pick $\sigma\kappa'$ as k, then

STS1: $(\sigma\kappa', v_r) \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_v$

STS2: $(\sigma\kappa', v_r') \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_v$.

For the first, we instantiate the IH 2 on the premise ($\diamond$) with step index $\sigma\kappa' + f_r + 1$ to obtain $(\sigma\kappa' + f_r + 1, \mathrm{L}(\theta)[y \mapsto v]e_2) \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_\varepsilon^{\sigma\kappa'}$.

By unrolling the definition of $\lfloor\!\lfloor \cdot \rfloor\!\rfloor_\varepsilon^{:}$ using $\sigma\kappa' < \sigma\kappa' + f_r + 1$, we get: $(\sigma\kappa' + 1, v_r) \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_v$.

By instantiating downward closure (Lemma 3) with $\sigma\kappa' \leq \sigma\kappa' + 1$, we get $(\sigma\kappa', v_r) \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_v$.

Next, we re-instantiate the IH 2 on the premise ($\diamond\diamond$) with step index $\sigma\kappa' + f_r' + 1$ to obtain $(\sigma\kappa' + f_r' + 1, \mathrm{R}(\theta)[y \mapsto v'']e_2) \in \lfloor\!\lfloor \sigma\tau \rfloor\!\rfloor_\varepsilon^{\sigma\kappa'}$.

By unrolling the definition of $\lfloor\!\lfloor \cdot \rfloor\!\rfloor_\varepsilon^{:}$ with $\sigma\kappa' < \sigma\kappa' + f_r' + 1$, we get:

k) $f'_r \leq \sigma\kappa'$

l) $(\sigma\kappa' + 1, v'_r) \in \ (\!|\sigma\tau|\!)_v$

Hence we conclude the statement 5. by instantiating downward closure (Lemma 3) on l) using $\sigma\kappa' \leq \sigma\kappa' + 1$.

The statement 4. follows by d) and k), $c' + f'_r + c_{case}(\mathbb{S}, \mathbb{C}) \leq \sigma\kappa_e + \sigma\kappa' + c_{case}(\mathbb{S}, \mathbb{C})$

subsubcase 2: $(m - f_e, \texttt{inl w}) \in \ [\![\sigma\tau_1 + \sigma\tau_2]\!]_{\mathtt{v}}$ and $v' = \texttt{inl}\ v''$ for some v''.

By unrolling the definition, we get $(m - f_e, \mathtt{w}) \in \ [\![\sigma\tau_1]\!]_{\mathtt{v}}$ ($\diamond$). Next, we instantiate the IH 3 on the premise $\Delta; \Phi; \Gamma, x : \tau_1 \vdash_{\epsilon \sqcup \mu} e_1 : \tau \mid \kappa'$ using $\epsilon \sqcup \mu = \mathbb{C}$ and $(m - f_e, \theta[x \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma\Gamma, \mathtt{x} : \sigma\tau_1]\!]$, which holds because

* $(m - f_e, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $m - f_e \leq m$,
* by ($\diamond$)

we have $(m - f_e, \theta[x \mapsto \mathtt{w}]^{\ulcorner}\mathtt{e_1}^{\urcorner}) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa'}$ ($\diamond\diamond$).

The rest of the proof follows the structure of the subcase where $\mu = \mathbb{S}$.

subsubcase 3: $\mathtt{w}' = \mathtt{new}(\mathtt{inl\ v}, \mathtt{inl\ v''})$

The proof of this case is symmetric to the case where we have $\mathtt{w}' = \mathtt{new}(\mathtt{inl\ v}, \mathtt{inr\ v''})$.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : \tau\{I/t\} \mid \kappa \qquad \Delta \vdash I :: S}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \mathtt{pack}\ e : (\exists t{::}S.\ \tau)^{\mathbb{S}} \mid \kappa}\ \exists\mathbf{I}$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner}\mathtt{pack}\ e^{\urcorner}) \in [\![(\exists t{::}S.\ \sigma\tau)^{\mathbb{S}}]\!]_{\varepsilon}^{\sigma\kappa}$.

Following the definition of $[\![\cdot]\!]_{\varepsilon}^{:}$, assume that

$$\frac{\mathrm{L}(\theta^{\ulcorner}e^{\urcorner}) \Downarrow T, f_e (\star) \qquad v = \mathtt{V}(T)}{\mathrm{L}(\theta^{\ulcorner}\mathtt{pack}\ e^{\urcorner}) \Downarrow \langle \mathtt{pack}\ v, \mathtt{pack}\ T \rangle, f_e + c_{pack}()}\ \mathbf{pack}$$

where $f = f_e + c_{pack}() < m$.

By Lemma 1, $\vdash \sigma I :: S$ ($\dagger$).

By IH 1 on the first premise, we get $(m, \theta^{\ulcorner}e^{\urcorner}) \in [\![\sigma\tau\{\sigma I/t\}]\!]_{\varepsilon}^{\sigma\kappa'}$.

By unfolding its definition using ($\star$) with the definition $f_e \leq f < m$, we get

a) $\langle T, \theta^{\ulcorner}e^{\urcorner} \rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$

b) $\mathrm{R}(\theta^{\ulcorner}e^{\urcorner}) \Downarrow T', f'$

c) $v = \mathrm{L}(\mathtt{w}') \ \wedge \ \mathtt{V}(\mathtt{T}') = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$

d) $c' \leq \sigma\kappa$

e) $(m - f_e, \mathtt{w}') \in [\![\sigma\tau\{\sigma\mathtt{I}/\mathtt{t}\}]\!]_{\mathtt{v}}$

Then, we can conclude as follows:

1. By a)
$$\frac{\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}' \qquad \mathtt{V}(T') = v'}{\langle\langle \_, \mathtt{pack}\ T\rangle, \theta^{\ulcorner}\mathtt{pack}\ e^{\urcorner}\rangle \curvearrowright \mathtt{pack}\ \mathtt{w}', \langle \mathtt{pack}\ \mathtt{v}', \mathtt{pack}\ \mathtt{T}'\rangle, \mathtt{c}'}\ \textbf{r-pack}$$
2. By b)
$$\frac{\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f' \qquad \mathtt{V}(T') = v'}{\mathrm{R}(\theta^{\ulcorner}\mathtt{pack}\ e^{\urcorner}) \Downarrow \langle \mathtt{pack}\ v', \mathtt{pack}\ T'\rangle, f' + c_{pack}()}\ \textbf{pack}$$
3. By c), we can conclude that $\mathtt{pack}\ v = \mathrm{L}(\mathtt{pack}\ \mathtt{w}') \ \wedge\ \mathtt{pack}\ \mathtt{v}' = \mathrm{R}(\mathtt{pack}\ \mathtt{w}')$
4. follows by d)
5. TS: $(m - f - c_{pack}(), \mathtt{pack}\ \mathtt{w}') \in [\![(\exists \mathtt{t}{::}\mathtt{S}.\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$.
   We know (†). RTS: $(m - f - c_{pack}(), \mathtt{w}') \in [\![\sigma\tau\{\sigma\mathtt{I}/\mathtt{t}\}]\!]_{\mathtt{v}}$. This follows by applying Lemma 3 to e) using $m - f - c_{pack}() \leq m - f$.

Case $$\frac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : (\exists t{::}S.\ \tau)^{\mu} \mid \kappa_e \qquad t :: S, \Delta; \Phi; x : \tau, \Gamma \vdash_{\mu} e' : \tau' \mid \kappa' \quad \models \mu \trianglelefteq \tau' \qquad t \notin FV(\Phi; \Gamma, \tau', \kappa') \qquad \kappa = \kappa_e + \kappa' + (((\epsilon \sqcup \mu) \doteq \mathbb{C})\ ?\ c_{unpack}(\epsilon, \mu)\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e' : \tau' \mid \kappa}\ \exists\textbf{E}$$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.
TS: $(m, \theta^{\ulcorner}\mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e'\mu^{\urcorner}) \in [\![\tau']\!]^{\sigma\kappa}_{\varepsilon}$.
Following the definition of $[\![\cdot]\!]^{\cdot}_{\varepsilon}$, assume that
$$\frac{\mathrm{L}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T, f_e(\star) \qquad \mathtt{V}(T) = \mathtt{pack}\ v \qquad \mathrm{L}(\theta^{\ulcorner} e'^{\urcorner}[v/x]) \Downarrow T_r, f_r(\star\star) \qquad \mathtt{V}(T_r) = v_r}{\mathrm{L}(\theta^{\ulcorner}\mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e'^{\urcorner}) \Downarrow \langle v_r, \mathtt{unpack}(T, x, T_r)\rangle, f_e + f_r + c_{unpack}(\mathbb{C}, \mathbb{S})}\ \textbf{unpack}$$
where $f = f_e + f_r + c_{unpack}(\mathbb{C}, \mathbb{S}) < m$ By IH 1 on the first premise, we get
$(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![(\exists t{::}S.\ \sigma\tau)^{\mu}]\!]^{\sigma\kappa_e}_{\varepsilon}$.
By unfolding its definition using ($\star$) with the definition $f_e \leq f < m$, we get

a) $\langle T, \theta^{\ulcorner} e^{\urcorner}\rangle \curvearrowright \mathtt{w}', T', c'$
b) $\mathrm{R}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow T', f'$
c) $v = \mathrm{L}(\mathtt{w}') \ \wedge\ \mathtt{V}(\mathtt{T}') = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$
d) $c' \leq \sigma\kappa$
e) $(m - f_e, \mathtt{w}') \in [\![(\exists t{::}S.\ \sigma\tau)^{\mu}]\!]_v$

There are two cases:

subcase 1: $\mu = \mathbb{S}$
We have $(m - f_e, \mathtt{pack}\ \mathtt{w}'') \in [\![(\exists \mathtt{t}{::}\mathtt{S}.\ \sigma\tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$.
By unrolling its definition, we get $\vdash I :: S$ (†) and $(m - f_e, \mathtt{w}'') \in [\![\sigma\tau\{\mathtt{I}/\mathtt{t}\}]\!]_{\mathtt{v}}$ (††).
By IH 1 on the second premise using

- $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$ using (†)

- $(m - f_e, \theta[x \mapsto \mathtt{w}'']), \in \mathcal{G}[\![\sigma[\mathtt{t} \mapsto \mathtt{I}](\Gamma, \mathtt{x} : \tau)]\!]$, which holds because
  - $(m - f_e, \theta) \in \mathcal{G}[\![\sigma[t \mapsto I]\Gamma]\!] = \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ and $m - f_e \leq m$, and noting that $t \notin FV(\Gamma)$
  - $(m - f_e, \mathtt{w}'') \in [\![\sigma[\mathtt{t} \mapsto \mathtt{I}]\tau]\!]_{\mathtt{v}}$ by (††)

we get $(m - f_e, \theta[x \mapsto \mathtt{w}'']^{\ulcorner}\mathtt{e}'^{\urcorner}) \in [\![\sigma[\mathtt{t} \mapsto \mathtt{I}]\tau']\!]_{\varepsilon}^{\sigma[\mathtt{t} \mapsto \mathtt{I}]\kappa'} = [\![\sigma\tau']\!]_{\varepsilon}^{\sigma\kappa'} =$ since $t \notin FV(\Phi; \Gamma, \tau', \kappa')$.

By unrolling its definition using (⋆⋆) with $f_r < m - f_e$, we get

f) $\langle T_r, \theta[x \mapsto \mathtt{w}'']^{\ulcorner}\mathtt{e}'^{\urcorner}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}}$

g) $\mathrm{R}([x \mapsto \mathtt{w}'']^{\ulcorner}\mathtt{e}'^{\urcorner}) \Downarrow \mathtt{T}'_{\mathtt{r}}, \mathtt{f}'_{\mathtt{r}}$

h) $v_r = \mathrm{L}(\mathtt{w}'_{\mathtt{r}}) \ \wedge \ \mathtt{V}(\mathtt{T}'_{\mathtt{r}}) = \mathtt{v}'_{\mathtt{r}} = \mathrm{R}(\mathtt{w}'_{\mathtt{r}})$

i) $c' \leq \sigma\kappa'$

j) $(m - f_e - f_r, \mathtt{w}'_{\mathtt{r}}) \in [\![\sigma\tau']\!]_{\mathtt{v}}$

We can conclude

1. By a) and f)
$$\frac{\langle T, \theta^{\ulcorner}e^{\urcorner}\rangle \curvearrowright \mathtt{pack}\ \mathtt{w}'', \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \theta^{\ulcorner}e'^{\urcorner}[\mathtt{w}''/\mathtt{x}]\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}} \qquad \mathtt{V}(T'_r) = v'_r}{\begin{array}{c}\langle\langle \_, \mathtt{unpack}(T, x, T_r)\rangle, \theta^{\ulcorner}\mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e'^{\urcorner}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \\ \langle v'_r, \mathtt{unpack}(T', x, T'_r)\rangle, c' + c'_r\end{array}}\ \textbf{r-unpack1}$$
2. By b) and g)
$$\frac{\mathrm{R}(\theta^{\ulcorner}e^{\urcorner}) \Downarrow T', f'(\star) \qquad \mathtt{V}(T') = \mathtt{pack}\ v'' \qquad \mathrm{R}(\theta^{\ulcorner}e'^{\urcorner}[v''/x]) \Downarrow T'_r, f'_r(\star\star) \qquad \mathtt{V}(T'_r) = v'_r}{\mathrm{R}(\theta^{\ulcorner}\mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e'\mu^{\urcorner}) \Downarrow v_r, \langle v_r, \mathtt{unpack}(T', x, T'_r)\rangle f' + f'_r + c_{unpack}(\mathbb{C}, \mathbb{S})}\ \textbf{unpack}$$
3. follows by h)
4. By d) and c), we get $c' + c'_r \leq \sigma(\kappa + \kappa')$
5. By Lemma 3 on j), we get $(m - f_e, \mathtt{w}'_{\mathtt{r}}) \in [\![\sigma\tau']\!]_{\mathtt{v}}$

subcase 2: $\mu = \mathbb{C}$

We have $(m - f_e, \mathtt{w}') \in [\![(\exists \mathtt{t}::\mathtt{S}.\ \sigma\tau)^{\mathbb{C}}]\!]_{\mathtt{v}}$.

There are two cases for $\mathtt{w}'$

- $\mathtt{w}' = \mathtt{pack}\ \mathtt{w}''$ s.t. $(m - f_e, \mathtt{pack}\ \mathtt{w}'') \in [\![(\exists \mathtt{t}::\mathtt{S}.\ \sigma\tau)^{\mathbb{C}}]\!]_{\mathtt{v}}$.
  By unrolling its definition, we get $\vdash I :: S$ (†) and $(m - f_e, \mathtt{w}') \in [\![\sigma\tau\{\mathtt{I}/\mathtt{t}\}]\!]_{\mathtt{v}}$ (††).
  By IH 1 on the second premise using
  - $\sigma[t \mapsto I] \in \mathcal{D}[\![t :: S, \Delta]\!]$ using (†)
  - $(m - f_e, \theta[x \mapsto \mathtt{w}']), \in \mathcal{G}[\![\sigma[\mathtt{t} \mapsto \mathtt{I}](\Gamma, \mathtt{x} : \tau)]\!]$, which holds because
    - $(m - f_e, \theta) \in \mathcal{G}[\![\sigma[t \mapsto I]\Gamma]\!] = \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ and $m - f_e \leq m$, and noting that $t \notin FV(\Gamma)$
    - $(m - f_e, \mathtt{w}') \in [\![\sigma[\mathtt{t} \mapsto \mathtt{I}]\tau]\!]_{\mathtt{v}}$ by (††)

we get $(m - f_e, \theta[x \mapsto \mathtt{w}']^{\ulcorner}\mathtt{e}'^{\urcorner}) \in [\![\sigma[\mathtt{t} \mapsto \mathtt{I}]\tau']\!]_{\varepsilon}^{\sigma[\mathtt{t}\mapsto\mathtt{I}]\kappa'} = [\![\sigma\tau']\!]_{\varepsilon}^{\sigma\kappa'} =$ since $t \notin FV(\Phi; \Gamma, \tau', \kappa')$.

By unrolling its definition using ($\star\star$) with $f_r < m - f_e$, we get

k) $\langle T_r, \theta[x \mapsto \mathtt{w}']^{\ulcorner}\mathtt{e}'^{\urcorner}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}}$

l) $\mathrm{R}([x \mapsto \mathtt{w}']^{\ulcorner}\mathtt{e}'^{\urcorner}) \Downarrow \mathtt{T}'_{\mathtt{r}}, \mathtt{f}'_{\mathtt{r}}$

m) $v_r = \mathrm{L}(\mathtt{w}'_{\mathtt{r}}) \ \wedge \ \mathtt{V}(\mathtt{T}'_{\mathtt{r}}) = \mathtt{v}'_{\mathtt{r}} = \mathrm{R}(\mathtt{w}'_{\mathtt{r}})$

n) $c' \leq \sigma\kappa'$

o) $(m - f_e - f_r, \mathtt{w}'_{\mathtt{r}}) \in [\![\sigma\tau']\!]_{\mathtt{v}}$

We can conclude

1. By a) and f)
$$\frac{\langle T, \theta^{\ulcorner}e^{\urcorner}\rangle \curvearrowright \mathtt{pack}\ \mathtt{w}', \mathtt{T}', \mathtt{c}' \qquad \langle T_r, \theta^{\ulcorner}e'^{\urcorner}[\mathtt{w}'/\mathtt{x}]\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \mathtt{T}'_{\mathtt{r}}, \mathtt{c}'_{\mathtt{r}} \qquad \mathtt{V}(T'_r) = v'_r}{\begin{array}{c}\langle\langle \_, \mathtt{unpack}(T, x, T_r)\rangle, \theta^{\ulcorner}\mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e'^{\urcorner}\rangle \curvearrowright \mathtt{w}'_{\mathtt{r}}, \\ \langle v'_r, \mathtt{unpack}(T', x, T'_r)\rangle, c' + c'_r\end{array}}\ \textbf{r-unpack1}$$
2. By b) and g)
$$\frac{\mathrm{R}(\theta^{\ulcorner}e^{\urcorner}) \Downarrow \mathtt{pack}\ v, Tf(\star) \qquad \mathrm{R}(\theta^{\ulcorner}e'^{\urcorner}[v/x]) \Downarrow v_r, T_r f_r(\star\star)}{\mathrm{R}(\theta^{\ulcorner}\mathtt{unpack}\ e\ \mathtt{as}\ x\ \mathtt{in}\ e'\mu^{\urcorner}) \Downarrow v_r, \mathtt{unpack}(T, x, T_r)v_r f + f_r + 1}\ \textbf{unpack}$$
3. follows by h)
4. By d) and c), we get $c' + c'_r \leq \sigma(\kappa + \kappa')$
5. By Lemma 3 on j), we get $(m - j, \mathtt{w}'_{\mathtt{r}}) \in [\![\sigma\tau']\!]_{\mathtt{v}}$

- $\mathtt{w}' = \mathtt{new}(\_, \mathtt{pack}\ \mathtt{v}')$

  This proof case is also similar to the rules like `case`, `app` etc.

Case $\dfrac{\Delta; \Phi \models C \qquad \Delta; \Phi \wedge C; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : (C \ \& \ \tau)^{\mathbb{S}} \mid \kappa}$ **c-andI**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner}e^{\urcorner}) \in [\![(\sigma C \ \& \ \sigma\tau)^{\mathbb{S}}]\!]_{\varepsilon}^{\sigma\kappa}$.

Following the definition of $[\![\cdot]\!]_{\varepsilon}^{\cdot}$, assume that

$\mathrm{L}(\theta^{\ulcorner}e^{\urcorner}) \Downarrow T, f(\star)$ and $f < m$

$\Phi \models \sigma C$ ($\dagger$).

By main assumption $\models \sigma\Phi$ and using the premise $\Delta; \Phi \models C$, combined with Assumption 12, we obtain $\models \sigma(\Phi \wedge C)$ ($\dagger$).

By IH 1 on the second premise using ($\dagger$), we get $(m, \theta^{\ulcorner}e^{\urcorner}) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa}$.

By unfolding its definition using ($\star$) and $f < m$, we get

a) $\langle T, \theta^{\ulcorner}e^{\urcorner}\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$

b) $\mathrm{R}(\theta^{\ulcorner}e^{\urcorner}) \Downarrow \mathtt{V}(T') = v', T'f'$

c) $v = \mathrm{L}(\mathtt{w}') \ \wedge \ \mathtt{v}' = \mathrm{R}(\mathtt{w}')$

d) $c' \leq \sigma\kappa$

e) $(m - f, \mathtt{w}') \in [\![\sigma\tau]\!]_{\mathtt{v}}$

Then, we can conclude as follows:

1. follows immediately from a)
2. follows immediately from b)
3. follows immediately from c)
4. follows immediately from d)
5. TS: $(m - f, \mathtt{w}') \in [\![(\mathtt{C}\ \&\ \tau)^{\mathbb{S}}]\!]_{\mathtt{v}}$.
   This follows by $\models \sigma C$ (obtained by (†)), and e).

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa' \qquad \forall x \in \Gamma \quad \Delta; \Phi \wedge C \models \Gamma(x) \sqsubseteq \Box(\Gamma(x)) \qquad \Delta; \Phi \wedge \neg C \models \kappa' \leq \kappa}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa}$ **r-split**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.
TS: $(m, \theta^{\ulcorner} e^{\urcorner}) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa}$.
Instead, we first show soundness of the following more general rule **split** and then derive **r-split** rule using **nochange** and **split** rules.

$$\frac{\Delta; \Phi \wedge C; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa \qquad \Delta; \Phi \wedge \neg C; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa}\ \textbf{split}$$

There are two cases:

subcase 1: $\models \sigma\Phi \wedge C$

Follows immediately by IH on the first premise.

subcase 2: $\models \sigma\Phi \wedge \neg C$

Follows immediately by IH on the second premise.

Next, we derive the **r-split** rule: assuming (⋆), (◇) and (†), we obtain $\Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa$.
Since the contexts $\Delta; \Phi$ stay same across derivations, we omit them for brevity.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa'(\star)}{C; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa'}\ \textbf{constr-weak} \qquad \forall x \in dom(\Gamma)\ \ C \models \Gamma(x) \sqsubseteq \square(\Gamma(x))(\diamond)}{C; \Gamma \vdash_{\mathbb{S}} e : \square(\tau) \mid 0}\ \textbf{nochange}}{C; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa}\ \sqsubseteq \qquad \dfrac{\dfrac{\Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa'(\star)}{\neg C; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa'}\ \textbf{weak} \qquad \neg C \models \kappa' \leq \kappa(\dagger)}{\neg C; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa}\ \sqsubseteq}{\vdash_{\mathbb{S}} e : \tau \mid \kappa}\ \textbf{split}$$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa' \qquad \forall x \in \Gamma \quad \Delta; \Phi \models \Gamma(x) \sqsubseteq \square(\Gamma(x)) \qquad \kappa = ((\epsilon = \mathbb{S}\ ?\ 0 :\ \kappa'))}{\Delta; \Phi; \Gamma, \Gamma' \vdash_{\mathbb{S}} e : \square(\tau) \mid \kappa}\ \textbf{nochange}$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma, \sigma\Gamma']\!]$ and $\models \sigma\Phi$. Note that $\epsilon = \mathbb{S}$, then $\sigma\kappa = 0$.
Let $\theta = \theta_1 \cup \theta_2$ where $(m, \theta_1) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $(m, \theta_2) \in \mathcal{G}[\![\sigma\Gamma']\!]$.
TS: $(m, \theta^{\ulcorner} e^{\urcorner}) \in\ [\![\square(\sigma\tau)]\!]^0_\varepsilon$
STS: $(m, \theta_1{}^{\ulcorner} e^{\urcorner}) \in\ [\![\square(\sigma\tau)]\!]^0_\varepsilon$ since e doesn't have any free variables from Γ'.
Unrolling the definition of $[\![\cdot]\!]^{\cdot}_\varepsilon$, assume that $\mathrm{L}(\theta^{\ulcorner} e^{\urcorner}) \Downarrow v, Tf$ $(\star)$ where $f < m$.
By IH on premise with θ_1, we get $(m, \theta_1{}^{\ulcorner} e^{\urcorner}) \in\ [\![\sigma\tau]\!]^{\sigma\kappa}_\varepsilon$.
Unrolling this definition with $(\star)$, we get

a) $\langle T, \theta_1{}^{\ulcorner} e^{\urcorner} \rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$
b) $\mathrm{R}(\theta_1{}^{\ulcorner} e^{\urcorner}) \Downarrow T', f'$
c) $v = \mathrm{L}(\mathtt{w}')\ \wedge\ \mathtt{V}(\mathtt{T}') = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$
d) $c' \leq \sigma\kappa$
e) $(m - f, \mathtt{w}') \in\ [\![\sigma\tau]\!]_{\mathtt{v}}$

Then,

1. follows immediately from a)
2. follows immediately from b)
3. follows immediately from c)
4. $c' = 0$ as shown in 5.
5. TS: $(m - f, \mathtt{w}') \in\ [\![\square(\sigma\tau)]\!]_{\mathtt{v}}$.
   Using e) and Lemma 8 , STS: $\mathtt{stable}(\mathtt{w}')$.
   From Lemma 7 and the premises of the typing rule, we get $\mathtt{stable}(\theta_1{}^{\ulcorner} e^{\urcorner})$. Using Lemma 6 and a), we get $\mathtt{stable}(\mathtt{w}')$ and $c' = 0$.

Case $\dfrac{\Upsilon(\zeta) = \zeta : (B_1 \cdots B_n) \xrightarrow{\kappa'} B \qquad \Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e : (B_i)^{\mu_i} \mid \kappa_{e_i} \qquad \mu_1 \sqcup \cdots \sqcup \mu_n = \mu \qquad \kappa = (\sum_{i=1}^{n} \kappa_{e_i}) + \kappa' + c_{prim}(\mathbb{S}, n, \mu_1, \cdots, \mu_n)}{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} \zeta\ (e_1 \cdots e_n) : (B)^{\mu} \mid \kappa}$ **primApp**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner}\zeta\ e_1 \cdots e_n{}^{\urcorner}) \in [\![B]\!]_{\varepsilon}^{\sigma\kappa}$

Assume that

$$\frac{\mathrm{L}(\theta^{\ulcorner}e_i{}^{\urcorner}) \Downarrow T_i, f_i \qquad v_i = \mathtt{V}(T_i) \qquad \widehat{\zeta}(v_i \cdots v_n) = (f_r, v_r)}{\zeta\ \mathrm{L}(\theta^{\ulcorner}(e_1 \cdots e_n)^{\urcorner}) \Downarrow \langle v_r, \mathtt{primApp}(T_1 \cdots T_n, \zeta)\rangle, (\sum_{i=1}^{n} f_i) + f_r + c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S})}\ \textbf{primapp}$$

Assume $f = (\sum_{i=1}^{n} f_i) + f_r + c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S}) < m$

By IH on e_i, we get $(m, \theta^{\ulcorner}e_i{}^{\urcorner}) \in [\![(B_i)^{\mu_i}]\!]_{\varepsilon}^{\sigma\kappa_{e_i}}$.

Unrolling each of these with the corresponding premise ($\star_i$) and noting that $f_i \leq f < m$, we get

a) $\langle T_i, \theta^{\ulcorner}e_i{}^{\urcorner}\rangle \curvearrowright \mathtt{w}'_{\mathtt{i}}, \mathtt{T}'_{\mathtt{i}}, \mathtt{c}'_{\mathtt{i}}$

b) $\mathrm{R}(\theta^{\ulcorner}e^{\urcorner}) \Downarrow T'_i, f'$

c) $v = \mathrm{L}(\mathtt{w}') \ \wedge\ \mathtt{V}(\mathtt{T}'_{\mathtt{i}}) = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$

d) $c'_i \leq \sigma\kappa_{e_i}$

e) $(m - f_i, \mathtt{w}'_{\mathtt{i}}) \in [\![(\mathtt{B}_{\mathtt{i}})^{\mu_{\mathtt{i}}}]\!]_{\mathtt{v}}$

There are two cases depending on whether $\mathtt{stable}(\theta^{\ulcorner}\zeta\ e_1 \cdots e_n{}^{\urcorner})$ or not.

subcase 1: $\mathtt{stable}(\theta^{\ulcorner}\zeta\ e_1 \cdots e_n{}^{\urcorner})$

1. $\dfrac{\mathtt{stable}(\theta^{\ulcorner}\zeta\ e_1 \cdots e_n{}^{\urcorner})}{\langle\langle v_r, \mathtt{primApp}(T, \zeta)\rangle, \theta^{\ulcorner}\zeta\ e_1 \cdots e_n{}^{\urcorner}\rangle \curvearrowright {}^{\ulcorner}v_r{}^{\urcorner}, \langle v_r, \mathtt{primApp}(T, \zeta)\rangle, 0}$ **r-prim-s**
2. Since $\mathtt{stable}(\theta^{\ulcorner}\zeta\ e_1 \cdots e_n{}^{\urcorner})$, we also have $\mathtt{stable}(\theta^{\ulcorner}e_i{}^{\urcorner})$. So, $\mathrm{L}(\theta^{\ulcorner}e_i{}^{\urcorner}) = \mathrm{R}(\theta^{\ulcorner}e_i{}^{\urcorner})$. Hence, from the evaluation judgment for $\mathrm{L}(\theta^{\ulcorner}\zeta\ e_i{}^{\urcorner})$, we get:
$$\frac{\mathrm{R}(\theta^{\ulcorner}e_i{}^{\urcorner}) \Downarrow T_i, f \qquad v_i = \mathtt{V}(T_i) \qquad \widehat{\zeta}(v_i \cdots v_n) = (f_r, v_r)}{\mathrm{R}(\theta^{\ulcorner}\zeta\ e^{\urcorner}) \Downarrow \langle v_r, \mathtt{primApp}(T, \zeta)\rangle, (\sum_{i=1}^{n} f_i) + f_r + c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S})}\ \textbf{primapp}$$
3. $v_r = \mathrm{L}({}^{\ulcorner}v_r{}^{\urcorner}) \ \wedge\ v_r = \mathrm{R}({}^{\ulcorner}v_r{}^{\urcorner})$ by definition of ${}^{\ulcorner}\cdot{}^{\urcorner}$.
4. $0 \leq (\sum_{i=1}^{n} f_i) + f_r + c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S})$, trivially.
5. TS: $(m - f, {}^{\ulcorner}v_r{}^{\urcorner}) \in [\![(B)^{\mu}]\!]_v$ (†).
   From Assumption 17, using $(m - f_i, \mathtt{w}'_{\mathtt{i}}) \in [\![(\mathtt{B}_{\mathtt{i}})^{\mu_{\mathtt{i}}}]\!]_{\mathtt{v}}$ obtained by e), we derive that $(m - f_i, \mathtt{merge}(v_r, v_r)) \in [\![(B)^{\mu}]\!]_v$. Then, by instantiating Lemma 3 with $m - f \leq m - f_i$, we get (†).

subcase 2: $\neg\mathtt{stable}(\theta^{\ulcorner}\zeta\ e_1 \cdots e_n{}^{\urcorner})$

From Assumption 17 using $\zeta : (B_1 \cdots B_n) \xrightarrow{\kappa'} B$ and $(m - f_i, \mathtt{w}'_{\mathtt{i}}) \in [\![(\mathtt{B})^{\mu_{\mathtt{i}}}]\!]_{\mathtt{v}}$ obtained by

e), we derive that

f) $\widehat{\zeta}(\mathrm{R}(\mathtt{w}')) = (\mathtt{f}'_\mathtt{r}, \mathtt{v}'_\mathtt{r})$

g) $(m - f_i, \mathtt{merge}(v_r, v'_r)) \in [\![(B)^\mu]\!]_v$

h) $f'_r \leq \kappa$

1. By c) and f),
$$\frac{\langle T_i, \theta^\ulcorner e_i \urcorner\rangle \curvearrowright \mathtt{w}'_\mathtt{i}, \mathtt{T}'_\mathtt{i}, \mathtt{c}'_\mathtt{i} \qquad v'_i = \mathtt{V}(T'_i) \qquad (f'_r, v'_r) = \widehat{\zeta}(v'_i \cdots v'_n)}{\langle\langle v_r, \mathtt{primApp}(T, \zeta)\rangle, \theta^\ulcorner \zeta\ e^\urcorner\rangle \curvearrowright} \textbf{r-prim}$$
$$\mathtt{merge}(v_r, v'_r),\ \langle v'_r, \mathtt{primApp}(T', \zeta)\rangle,\ (\sum_{i=1}^{n} c'_i) + f'_r + c_{prim}(\mathbb{S}, n, \mathbb{C} \cdots \mathbb{C})$$

2. By using b), c) and f)
$$\frac{\mathrm{R}(\theta^\ulcorner e_i \urcorner) \Downarrow T'_i, f'_i \qquad v'_i = \mathtt{V}(T'_i) \qquad \widehat{\zeta}(v'_1 \cdots v'_n) = (f'_r, v'_r)}{\mathrm{R}(\theta^\ulcorner \zeta\ e^\urcorner) \Downarrow \langle v'_r, \mathtt{primApp}(T', \zeta)\rangle, (\sum_{i=1}^{n} f'_i) + f'_r + c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S})} \textbf{primapp}$$

3. follows immediately: $v_r = \mathrm{L}(\mathtt{merge}(v_r, v'_r))$ and $v'_r = \mathrm{R}(\mathtt{merge}(v_r, v'_r))$

4. From d) and h), $(\sum_{i=1}^{n} c'_i) + f'_r + c_{prim}(\mathbb{S}, n, \mathbb{C} \cdots \mathbb{C}) \leq \sigma\kappa_e + \kappa + c_{prim}(\mathbb{S}, n, \mu_1, \cdots, \mu_n)$ where $\mu_1 \sqcup \cdots \sqcup \mu_n = \mathbb{C}$. So, the LHS and RHS cost functions will have the same

5. From g) we get $(m - f_i, \mathtt{merge}(v_r, v'_r)) \in [\![(B)^\mu]\!]_v \equiv [\![(B)^\mathbb{C}]\!]_v$.
Then, by instantiating Lemma 3 with $m - f \leq m - f_i$, we get $(m - f, \mathtt{merge}(v_r, v'_r)) \in [\![(B)^\mu]\!]_v$.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_\mathbb{S} e : \tau' \mid \kappa' \qquad \models \tau' \sqsubseteq \tau \qquad \kappa' \leq \kappa}{\Delta; \Phi; \Gamma \vdash_\mathbb{S} e : \tau \mid \kappa}\ \sqsubseteq$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^\ulcorner e^\urcorner) \in [\![\sigma\tau]\!]^{\sigma\kappa}_\varepsilon$

Following the definition of $[\![\cdot]\!]^{\cdot}_\varepsilon$, assume that $\mathrm{L}(\theta^\ulcorner e^\urcorner) \Downarrow T, f$ ($\star$) where $f < m$.

By IH on premise, we get $(m, \theta^\ulcorner e^\urcorner) \in [\![\sigma\tau']\!]^{\sigma\kappa'}_\varepsilon$.

Unrolling this and using the assumption marked ($\star$), we get

a) $\langle T, \theta^\ulcorner e^\urcorner\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$

b) $\mathrm{R}(\theta^\ulcorner e^\urcorner) \Downarrow T', f'$

c) $v = \mathrm{L}(\mathtt{w}')\ \wedge\ \mathtt{V}(\mathtt{T}') = \mathtt{v}' = \mathrm{R}(\mathtt{w}')$

d) $c' \leq \sigma\kappa$

e) $(m - j, \mathtt{w}') \in [\![\sigma\tau']\!]_\mathtt{v}$

Then,

1. follows immediately from a)
2. follows immediately from b)
3. follows immediately from c)

4. Applying Assumption 12 to $\Delta; \Phi \models \kappa \leq \kappa'$ and the assumptions $\models \sigma\Phi$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$, we get $\sigma\kappa \leq \sigma\kappa'$. Therefore, using d), $c' \leq \sigma\kappa \leq \sigma\kappa'$.
5. Applying Lemma 11 using $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and e), we get $(m - f, \mathtt{w}') \in [\![\sigma\tau]\!]_{\mathtt{v}} \subseteq [\![\sigma\tau']\!]_{\mathtt{v}}$.

Case $\dfrac{\Delta; \Phi; f : \Box((\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}), x : \tau_1, \Gamma \vdash_\delta e : \tau_2 \mid \kappa \qquad \forall x \in \Gamma \quad \Delta; \Phi \models \Gamma(x) \sqsubseteq \Box(\Gamma(x))}{\Delta; \Phi; \Gamma, \Gamma' \vdash_{\mathbb{S}} \mathtt{fix}\ f(x).\, e : \Box((\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}) \mid 0}$ **fix2**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.
Let $\theta = \theta_1 \cup \theta_2$ where $(m, \theta_1) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $(m, \theta_2) \in \mathcal{G}[\![\sigma\Gamma']\!]$.
TS: $(m, \theta\ulcorner \mathtt{fix}\ f(x).\, e\urcorner) \in [\![\Box((\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}})]\!]^0_\varepsilon$.
STS: $(m, \theta_1\ulcorner \mathtt{fix}\ f(x).\, e\urcorner) \in [\![\Box((\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}})]\!]_v$ by Lemma 5 and $\forall z \in \Gamma', z \notin FV(e)$.
Let $F = \theta_1\ulcorner \mathtt{fix}\ f(x).\, e\urcorner$.
There are two cases.

subcase 1: $\delta = \mathbb{S}$

We prove the more general statement $\forall k \leq m.\ (k, \theta\ulcorner \mathtt{fix}\ f(x).\, e\urcorner) \in [\![\Box((\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}})]\!]_v$ by subinduction on k.

subsubcase 1: $k = 0$

Unfolding the definition of $[\![\cdot]\!]_v$ at box type, and unrolling the definition of $[\![\cdot]\!]_v$ at the function type, we only need to show that $\mathtt{stable}(F)$. This follows from Lemma 7 and the assumption $\forall y \in \Gamma.\ \Delta; \Phi \models \Gamma(y) \sqsubseteq \Box(\Gamma(y))$.

subsubcase 2: $k + 1 \leq m$

Assume, by the sub-IH, that $(k, F) \in [\![\Box(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)]\!]_v$, i.e. $(k, F) \in [\![\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2]\!]_v$($\star$) and $\mathtt{stable}(F)$ ($\star\star$).
STS: $(k + 1, F) \in [\![\Box(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)]\!]_v$
STS: $(k + 1, F) \in [\![\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2]\!]_v$ and $\mathtt{stable}(F)$.
By ($\star\star$), $\mathtt{stable}(F)$.
Following the definition $[\![\cdot]\!]_v$, pick $j < k + 1$.
Assume that $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$. Then, STS: $(j, \theta_1\ulcorner e\urcorner[F/f, \mathtt{w}/\mathtt{x}]) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_\varepsilon$ ($\star\star\star$).
Instantiate the IH 1 on the premise of the typing rule using:
$(j, \theta_1[f \mapsto F, x \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma(\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : \Box(\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \tau_2))]\!]$, which holds because:

- $(j, \theta_1) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta_1) \in \mathcal{G}[\![\Gamma]\!]$ and $j \leq m$,
- $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$
- $(j, F) \in [\![\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2]\!]_v$ by Lemma 3 on ($\star$) and $j \leq k$

We immediately get $(j, \theta_1[f \mapsto F, x \mapsto \mathtt{w}]\ulcorner \mathtt{e}\urcorner) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_\varepsilon$, which is same as ($\star\star\star$).

subcase 2: $\delta = \mathbb{C}$

By unrolling the definition, TS: $(m, \theta_1\ulcorner \mathtt{fix}\ f(x).\, e\urcorner) \in [\![\sigma\tau_1 \xrightarrow{\mathbb{C}(\kappa)} \sigma\tau_2]\!]_v$ and $\mathtt{stable}(F)$.

- **STS1:** $\forall k.\ (k, \mathrm{L}(F)) \in \ (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$ and $(k, \mathrm{R}(F)) \in \ (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$ and $\mathtt{stable}(F)$.

  Proof proceeds by sub-induction on k.

  i. **case** $k = 0$

  Unrolling the definition of $[\![\cdot]\!]_v$ at the function type, we only need to show that $\mathtt{stable}(F)$. This follows from Lemma 7 and the assumption $\forall y \in \Gamma.\ \Delta; \Phi \models \Gamma(y) \sqsubseteq \square(\Gamma(y))$.

  ii. **case** $k + 1$

  Assume by sub-IH that $(k, \mathrm{L}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$(†) and $(k, \mathrm{R}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$(†) and $\mathtt{stable}(F)$ (†††).

  STS1: $(k+1, \mathrm{L}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$.

  STS2: $(k+1, \mathrm{R}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$.

  $\mathtt{stable}(F)$ is immediately shown by (†††).

  We first show the first statement.

  Pick $j < k{+}1$ s.t. $(j, v) \in \ (\!|\sigma\tau_1|\!)_v$. Then, STS: $(j, \mathrm{L}(\theta^{\ulcorner} e^{\urcorner})[\mathrm{L}(F)/f, v/x]) \in \ (\!|\tau_2|\!)^{\sigma\kappa}_{\varepsilon}$(†††)

  Instantiate the IH 2 on the premise of the typing rule using $\delta = \mathbb{C}$ and $(j, \mathrm{L}(\theta_1)^{\ulcorner} e^{\urcorner}[f \mapsto \mathrm{L}(F), x \mapsto v]) \in \mathcal{G}(\!|\Gamma, x : \tau_1, f : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)$, which holds because:

  $*$ $(j, \mathrm{L}(\theta_1)) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by instantiating Lemma 2 with j using $(m, \theta_1) \in \mathcal{G}[\![\Gamma]\!]$,

  $*$ $(j, v) \in \ (\!|\sigma\tau_1|\!)_v$

  $*$ $(j, \mathrm{L}(F)) \in \ [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on (†) using $j \leq k$

  We immediately get $(j, \mathrm{L}(\theta)[f \mapsto \mathrm{L}(F), x \mapsto v]\mathrm{L}(^{\ulcorner} e^{\urcorner})) \in \ (\!|\sigma\tau_2|\!)^{\sigma\kappa}_{\varepsilon}$.

  Next, we show the second statement.

  Pick $j < k{+}1$ s.t. $(j, v) \in \ (\!|\sigma\tau_1|\!)_v$. Then, STS: $(j, \mathrm{R}(\theta^{\ulcorner} e^{\urcorner})[\mathrm{R}(F)/f, v/x]) \in \ (\!|\tau_2|\!)^{\sigma\kappa}_{\varepsilon}$(†††)

  Instantiate the IH 2 on the premise of the typing rule using $\delta = \mathbb{C}$ and $(j, \mathrm{R}(\theta_1)^{\ulcorner} e^{\urcorner}[f \mapsto \mathrm{R}(F), x \mapsto v]) \in \mathcal{G}(\!|\Gamma, x : \tau_1, f : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)$, which holds because:

  $*$ $(j, \mathrm{R}(\theta_1)) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by instantiating Lemma 2 with j using $(m, \theta_1) \in \mathcal{G}[\![\Gamma]\!]$,

  $*$ $(j, v) \in \ (\!|\sigma\tau_1|\!)_v$

  $*$ $(j, \mathrm{R}(F)) \in \ [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on (†) using $j \leq k$

  We immediately get $(j, \mathrm{R}(\theta)[f \mapsto \mathrm{R}(F), x \mapsto v]\mathrm{R}(^{\ulcorner} e^{\urcorner})) \in \ (\!|\sigma\tau_2|\!)^{\sigma\kappa}_{\varepsilon}$.

- **STS 2:** $\forall j < m.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, {}^{\ulcorner}\mathtt{e}^{\urcorner}[\mathtt{F/f}][\mathtt{w/x}]) \in [\![\tau_2]\!]^{\kappa}_{\varepsilon}$ and $\mathtt{stable}(F)$

  Proof by sub-induction on m.

  i. **case** $m = 0$ Since there exists no positive $\mathtt{j} < 0$, we only need to show $\mathtt{stable}(f)$, which follows from Lemma 7 and the assumption $\forall y \in \Gamma.\ \Delta; \Phi \models \Gamma(y) \sqsubseteq \square(\Gamma(y))$.

  ii. **case** $m = m' + 1$

  STS: $\forall j < m' + 1.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta^{\ulcorner}\mathtt{e}^{\urcorner}[\mathtt{F/f}][\mathtt{w/x}]) \in [\![\tau_2]\!]^{\kappa}_{\varepsilon}$ and $\mathtt{stable}(F)$.

  There are two possible cases.

– $j < m'$

Then, by sub-IH, we know that $\forall j < m'.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta\ulcorner \mathtt{e} \urcorner[\mathtt{F/f}][\mathtt{w/x}]) \in [\![\tau_2]\!]^{\kappa}_{\varepsilon}$ and $\mathtt{stable}(F)$. Since $j < m' < m' + 1$, we can immediately conclude.

– $j = m'$

Since $j = m' < m' + 1$, we assume that $(m', \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}}(\diamond)$.

STS: $(m', \theta\ulcorner e \urcorner[F/f][\mathtt{w/x}]) \in [\![\tau_2]\!]^{\kappa}_{\varepsilon}(\diamond\diamond)$

By IH 3 on the premise of the typing rule using $(m', \theta_1\ulcorner e \urcorner[f \mapsto F, x \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : \Box(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)]\!]$, which holds because:

* $(m', \theta_1) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by instantiating Lemma 3 with $m' < m$ and $(m, \theta_1) \in \mathcal{G}[\![\sigma\Gamma]\!]$
* $(m', \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$ by $(\diamond)$
* $(m', F) \in [\![\Box(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)]\!]_v$ which is obtained as shown in (STS 1) above

We immediately get $(m', \theta_1[f \mapsto F, x \mapsto \mathtt{w}]\ulcorner \mathtt{e} \urcorner) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$, which is same as $(\diamond\diamond)$.

Proof of statement (2):

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.

Case $\dfrac{}{\Delta; \Phi; \Gamma, x : \tau \vdash_{\mathbb{C}} x : \tau \mid c_{var}()}$ **var**

Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau|\!)$ and $\models \sigma\Phi$. Note that $\epsilon = \mathbb{C}$.

TS: $(m, \mathcal{U}(x)) \in (\!|\sigma\tau|\!)^{c_{var}()}_{\varepsilon}$.

By Value Lemma (Lemma 5) and cost weakening, STS: $(m, \mathcal{U}(x)) \in (\!|\sigma\tau|\!)_v$.

This follows from the definition of $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau|\!)$.

Case $\dfrac{}{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \mathtt{r} : (\mathtt{real})^{\mathbb{S}} \mid c_{real}()}$ **real**

Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$. Note that $\epsilon = \mathbb{C}$.

TS: $(m, \mathcal{U}(\mathtt{r})) \in (\!|(\mathtt{real})^{\mathbb{S}}|\!)^{c_{real}()}_{\varepsilon}$.

Assume that $c_{real}() < m$. By unrolling the definition, we can immediately show:

1. $\dfrac{}{\mathtt{r} \Downarrow \langle \mathtt{r}, \mathtt{r} \rangle, c_{real}()}$ **r**
2. $(m - c_{real}(), \mathtt{r}) \in [\![(\mathtt{real})^{\mathbb{S}}]\!]_v$
3. $c_{real}() \leq c_{real}()$

Case $\dfrac{}{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \mathtt{nil} : (\mathtt{list}\ [0]^0\ \tau)^{\mathbb{S}} \mid c_{nil}()}$ **nil**

Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

Assume that $c_{nil}() < m$. By unrolling the definition, we can immediately show:

1. $\dfrac{}{\texttt{nil} \Downarrow \langle \texttt{nil}, \texttt{nil} \rangle, c_{nil}()}$ **nil**
2. $(k - c_{nil}(), \texttt{nil}) \in [\![(\texttt{list}\,[0]^0\ \sigma\tau)^{\mathbb{S}}]\!]_v$
3. $c_{nil}() < c_{nil}()$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e_1 : \square(\tau) \mid \kappa_1 \qquad \Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e_2 : (\texttt{list}\,[n]^\alpha\ \tau)^\mu \mid \kappa_2}{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \texttt{cons}(e_1,\, e_2) : (\texttt{list}\,[n+1]^\alpha\ \tau)^\mu \mid \kappa_1 + \kappa_2 + c_{cons}()}$ **cons**

Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \mathcal{U}(\texttt{cons}(e_1,\, e_2))) \in (\!|(\texttt{list}\,[\sigma n + 1]^{\sigma\alpha}\ \tau)^\mu|\!)_\varepsilon^{\sigma(\kappa_1+\kappa_2)+c_{cons}()}$.
Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that $\sigma(\kappa_1 + \kappa_2) + c_{cons}() < m$. STS:

1. $\mathcal{U}(\texttt{cons}(e_1,\, e_2)) \Downarrow \langle v_r, D_r \rangle, f_r$
2. $(m - f_r, v_r) \in (\!|(\texttt{list}\,[\sigma n + 1]^{\sigma\alpha}\ \tau)^\mu|\!)_v$
3. $f_r \leq \sigma(\kappa_1 + \kappa_2) + c_{cons}()$.

By IH 2 on e_1, we get $(m, \mathcal{U}(e_1)) \in (\!|\square(\sigma\tau)|\!)_\varepsilon^{\sigma\kappa_1}$. Unrolling its definition using $\sigma\kappa_1 < m$, we get

a) $\mathcal{U}(e_1) \Downarrow \langle v_1, D_1 \rangle, f_1$
b) $(m - f_1, v_1) \in (\!|\square(\sigma\tau)|\!)_v$
c) $f_1 \leq \sigma\kappa_1$.

By IH 2 on e_1, we get $(m, \mathcal{U}(e_2)) \in (\!|(\texttt{list}\,[\sigma n]^{\sigma\alpha}\ \tau)^\mu|\!)_\varepsilon^{\sigma\kappa_2}$. Unrolling its definition using $\sigma\kappa_2 < m$, we get

d) $\mathcal{U}(e_2) \Downarrow \langle v_2, D_2 \rangle, f_2$
e) $(m - f_2, v_2) \in (\!|(\texttt{list}\,[\sigma n]^{\sigma\alpha}\ \tau)^\mu|\!)_v$
f) $f_2 \leq \sigma\kappa_2$.

Then, we can conclude

1. Using a) and c) and $T_i = \langle v_i, D_i \rangle$
$$\frac{\mathcal{U}(e_1) \Downarrow T_1, f_1 \qquad \mathcal{U}(e_2) \Downarrow T_2, f_2}{\mathcal{U}(\texttt{cons}(e_1,\, e_2)) \Downarrow \langle \texttt{cons}(v_1,\, v_2), \texttt{cons}(T_1, T_2) \rangle, f_1 + f_2 + c_{cons}()}\ \textbf{cons}$$
2. Next, we apply Lemma 3 to b) and e) respectively, and obtain: $(m-(f_1+f_2+c_{cons}()), v_1) \in (\!|\tau|\!)_v$ and $(m-(f_1+f_2+c_{cons}()), v_2) \in (\!|(\texttt{list}\,[\sigma n]^{\sigma\alpha}\ \tau)^\mu|\!)_v$ since $m-(f_1+f_2+c_{cons}()) \leq m - f_i$. Combining these two, we get $(m-(f_1+f_2+c_{cons}()), \texttt{cons}(v_1,\, v_2)) \in (\!|(\texttt{list}\,[\sigma n + 1]^{\sigma\alpha}\ \tau)^\mu|\!)_v$.

3. By combining c) and f), we obtain $f_1 + f_2 + c_{cons}() \leq \sigma\kappa_1 + \sigma\kappa_2 + c_{cons}()$.

Case $\dfrac{\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : \tau \mid \kappa_1 \qquad \Delta;\Phi;\Gamma \vdash_\epsilon e_2 : (\texttt{list}\,[n]^{\alpha-1}\ \tau)^{\mathbb{S}} \mid \kappa_2 \qquad \Delta;\Phi \models \alpha > 0}{\Delta;\Phi;\Gamma \vdash_\epsilon \texttt{cons}(e_1, e_2) : (\texttt{list}\,[n+1]^{\alpha}\ \tau)^{\mathbb{S}} \mid \kappa_1 + \kappa_2 + 1}$ **cons2**

Proof of this case is very similar to the case **cons1** since the unary relation doesn't take the modes and number of allowed changes α into account.

Case $\dfrac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_{\mathbb{C}} e : (\texttt{list}\,[n]^{\alpha}\ \tau)^{\mu} \mid \kappa_e \qquad \Delta;\Phi \wedge n \doteq 0 \wedge \alpha \doteq 0;\Gamma \vdash_{\mathbb{C}} e_1 : \tau' \mid \kappa' \\ i :: \iota, \Delta;\Phi \wedge n \doteq i+1; h : \square(\tau), tl : \texttt{list}\,[i]^{\alpha}\ \tau, \Gamma \vdash_{\mathbb{C}} e_2 : \tau' \mid \kappa' \\ \beta :: \iota, i :: \iota, \Delta;\Phi \wedge n \doteq i+1 \wedge \alpha \doteq \beta + 1; h : \tau, tl : \texttt{list}\,[i]^{\beta}\ \tau, \Gamma \vdash_{\mathbb{C}} e_2 : \tau' \mid \kappa' \\ \kappa = \kappa_e + \kappa' + c_{caseL}(\mathbb{C}, \mu)\end{array}}{\Delta;\Phi;\Gamma \vdash_{\mathbb{C}} \texttt{case}_{\texttt{L}}\ e\ \texttt{of nil}\ \rightarrow\ e_1 \mid \texttt{cons}(h, tl)\ \rightarrow\ e_2 : \tau' \mid \kappa}$ **caseL**

Assume that $(m,\mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m,\mathcal{U}(\texttt{case}_{\texttt{L}}\ e\ \texttt{of nil}\ \rightarrow\ e_1 \mid \texttt{cons}(h, tl)\ \rightarrow\ e_2)) \in (\!|\sigma\tau'|\!)^{\sigma\kappa}_{\varepsilon}$.
Unrolling the definition of $(\!|\cdot|\!)^{\cdot}_{\varepsilon}$, assume that $\sigma(\kappa_e + \kappa') + c_{caseL}(\mathbb{C}, \mu) < m$, STS:

1. $\mathcal{U}(\texttt{case}_{\texttt{L}}\ e\ \texttt{of nil}\ \rightarrow\ e_1 \mid \texttt{cons}(h, tl)\ \rightarrow\ e_2) \Downarrow \langle v_r, D_r\rangle, f_r$
2. $(m - f, v_r) \in (\!|\sigma\tau'|\!)_v$
3. $f_r \leq \sigma(\kappa_e + \kappa') + c_{caseL}(\mathbb{C}, \mu)$

By IH 2 on e and unrolling its definition with $\sigma\kappa_e < m$, we get

a) $\mathcal{U}(e) \Downarrow \langle v, D\rangle, f$
b) $(m - f_e, v) \in (\!|\texttt{list}\,[\sigma n]^{\sigma\alpha}\ \tau|\!)_v$
c) $f \leq \sigma\kappa_e$

There are two cases for a)

subcase 1: We have

d) $\mathcal{U}(e) \Downarrow \langle \texttt{nil}, D\rangle, f$
e) $(m - f_e, \texttt{nil}) \in (\!|\texttt{list}\,[0]^{\sigma\alpha}\ \sigma\tau|\!)_v$

Then, by IH 2 on e_1 using

- $\models \sigma\Phi \wedge \sigma n \doteq 0 \wedge \sigma\alpha \doteq 0$ obtained by combining $\models \sigma\Phi$ with $\models \sigma n \doteq 0$ and $\models \sigma\alpha \doteq 0$ by e)
- $(m,\mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$

and showing that $\sigma\kappa' < m$, we get

f) $\mathcal{U}(e_1) \Downarrow \langle v_1, D_1\rangle, f_1$
g) $(m - f_1, v_1) \in (\!|\sigma\tau'|\!)_v$
h) $f_1 \leq \sigma\kappa'$

Then, we can conclude by showing

1. By d) and f)
$$\frac{\mathcal{U}(e) \Downarrow T, f \qquad \mathtt{V}(T) = \mathtt{nil} \qquad \mathcal{U}(e_1) \Downarrow T_1, f_1 \qquad \mathtt{V}(T_1) = v_1}{\begin{array}{l}\mathcal{U}(\mathtt{case_L}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2) \Downarrow \\ \quad \langle v_1, \mathtt{case_{nil}}(T, T_1)\rangle, f + f_1 + c_{caseL}(\mathbb{C}, \mathbb{S})\end{array}}\ \textbf{case-nil}$$
2. By applying Lemma 3 to g), we get $(m - (f + f_1 + c_{caseL}(\mathbb{C}, \mathbb{S})), v_1) \in \ (\!|\sigma\tau'|\!)_v$ since $m - (f + f_1 + c_{caseL}(\mathbb{C}, \mathbb{S})) \leq m - f_1$.
3. By combining c) and h), we also obtain $f + f_1 + c_{caseL}(\mathbb{C}, \mathbb{S}) \leq \sigma\kappa_e + \sigma\kappa' + c_{caseL}(\mathbb{C}, \mu)$.

subcase 2: We have

a) $\mathcal{U}(e) \Downarrow \langle \mathtt{cons}(v_1,\ v_2), \mathtt{cons}(T_1, T_2)\rangle, f$
b) $(m - f_e, \mathtt{cons}(v_1,\ v_2)) \in \ (\!|\mathtt{list}\ [I + 1]^{\sigma\alpha}\ \sigma\tau|\!)_v$
c) $f \leq \sigma\kappa_e$

There are two cases, bu since unary relation doesn't take variations into account, we only show one of these cases:
by IH 2 on e_2 using

- $\sigma[i \mapsto I] \in \mathcal{D}[\![i :: \iota, \Delta]\!]$
- $\models \sigma[i \mapsto I](\Phi \wedge n \doteq i + 1)$ obtained by combining $\models \sigma\Phi$ with $\models \sigma n \doteq I + 1$ by b)
- $(m - f_e, \mathcal{U}[h \mapsto v_1, tl \mapsto v_2]) \in \mathcal{G}(\!|\sigma[i \mapsto I](\Gamma, h : \tau, tl \mapsto \mathtt{list}\ [i]^{-}\ \tau)|\!)$ by expanding b)

and assuming that $\sigma\kappa' < m$, we get

d) $\mathcal{U}(e_2)[v_1/h, v_2/tl] \Downarrow \langle v_r, D'_r\rangle, f_2$
e) $(m - f_e - f_2, v_r) \in \ (\!|\sigma\tau'|\!)_v$
f) $f_2 \leq \sigma\kappa'$

Then, we can conclude by

1. By a) and c)
$$\frac{\mathcal{U}(e) \Downarrow \langle \mathtt{cons}(v_1,\ v_2), \mathtt{cons}(T_1, T_2)\rangle, f \qquad \mathcal{U}(e_2)[v_1/h, v_2/tl] \Downarrow T_r, f_2 \qquad v_r = \mathtt{V}(T)_r}{\begin{array}{l}\mathcal{U}(\mathtt{case_L}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2) \Downarrow v_r. \\ \quad \langle v_r, \mathtt{case_{cons}}(\mathtt{cons}(T_1, T_2), T_r)\rangle, f + f_2 + c_{caseL}(\mathbb{C}, \mathbb{S})\end{array}}\ \textbf{case-cons}$$
2. By d) we get $(m - f_e - f_2 - c_{caseL}(\mathbb{C}, \mathbb{S}), v_r) \in \ (\!|\sigma\tau'|\!)_v$.

3. By combining b) and d), we also obtain $f_e + f_2 + c_{caseL}(\mathbb{C}, \mathbb{S}) \leq \sigma\kappa_e + \sigma\kappa' + c_{caseL}(\mathbb{C}, \mu)$.

Case $$\frac{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e : \tau \mid \kappa \qquad \forall x \in \Gamma \quad \Delta; \Phi \models \Gamma(x) \sqsubseteq \square(\Gamma(x))}{\Delta; \Phi; \Gamma, \Gamma' \vdash_{\mathbb{C}} e : \square(\tau) \mid \kappa}\ \textbf{nochange}$$

Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma, \sigma\Gamma'|\!)$ and $\models \sigma\Phi$.

Let $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ where $(m, \mathcal{U}_1) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $(m, \mathcal{U}_2) \in \mathcal{G}(\!|\sigma\Gamma'|\!)$.
TS: $(m, \mathcal{U}e) \in (\!|\Box(\sigma\tau)|\!)_\varepsilon^{\sigma\kappa}$
STS: $(m, \mathcal{U}_1 e) \in (\!|\Box(\sigma\tau)|\!)_\varepsilon^{\sigma\kappa}$ since e doesn't have any free variables from Γ'.
Assume that $\sigma\kappa < m$
By IH 2 on the premise using this assumption, we get

a) $\mathcal{U}e \Downarrow \langle v, D\rangle, f$
b) $(m - f, v) \in (\!|\sigma\tau|\!)_v$
c) $f \leq \sigma\kappa$

Then we can conclude

1. By a)
2. By b), we get $(m - f, v) \in (\!|\Box(\sigma\tau)|\!)_v = (\!|\sigma\tau|\!)_v$
3. $f \leq \sigma\kappa$.

Case $\dfrac{\Delta; \Phi; f : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}, x : \tau_1, \Gamma \vdash_\delta e : \tau_2 \mid \kappa}{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \texttt{fix}\ f(x).e : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}} \mid c_{fix}()}$ **fix1**

Assume that $(m, \mathcal{U}) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.
TS: $(m, \texttt{fix}\ f(x).\mathcal{U}e) \in (\!|(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)_\varepsilon^{c_{fix}()}$.
By unrolling the definition, assume that $c_{fix}() < m$
We can immediately show 1^{st} and 2^{nd}:

1. $\texttt{fix}\ f(x).\mathcal{U}e \Downarrow \langle \texttt{fix}\ f(x).\mathcal{U}e, \texttt{fix}\ f(x).\mathcal{U}e\rangle, c_{fix}()$ with **fix** evaluation rule
2. $c_{fix}() < c_{fix}()$

STS: $(m - c_{fix}(), \texttt{fix}\ f(x).\mathcal{U}e) \in (\!|(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)_v$
Let $F = \texttt{fix}\ f(x).\mathcal{U}e$.
There are two cases.

subcase 1: $\delta = \mathbb{S}$
By definition, any function is in $(\!|(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)_v$.

subcase 2: $\delta = \mathbb{C}$
We prove the more general statement
$\forall k \leq m - c_{fix}().\ (k, \texttt{fix}\ f(x).\mathcal{U}e) \in (\!|(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{C}}|\!)_v$ by subinduction on k.

subsubcase 1: $k = 0$ is vacuous from the definition $(\!|\cdot|\!)_v$ at the function type.

subsubcase 2: $k + 1 \leq m - c_{fix}()$
Assume, by the sub-IH, that $(k, F) \in (\!|(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)_v$ ($\star$)

STS: $(k+1, F) \in (\!|(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)_v$

Following the definition, pick $j < k + c_{fix}()$.

Assume that $(j, v) \in (\!|\sigma\tau_1|\!)_v$. Then, STS: $(j, \mathcal{U}e[F/f, \mathtt{w}/\mathtt{x}]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma\kappa}$ ($\star\star$).

Instantiate the IH 2 on the premise of the typing rule using:

$(j, \mathcal{U}[f \mapsto F, x \mapsto \mathtt{w}]) \in \mathcal{G}(\!|\sigma(\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : (\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \tau_2)^{\mathbb{S}})|\!)$, which holds because:

* $(j, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by Lemma 3 using $(m, \mathcal{U}) \in \mathcal{G}(\!|\Gamma|\!)$ and $j \leq m$,
* $(j, v) \in (\!|\sigma\tau_1|\!)_v$
* $(j, F) \in (\!|(\sigma\tau_1 \xrightarrow{\mathbb{C}(\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)_v$ by Lemma 3 on ($\star$) and $j \leq k$

We immediately get $(j, \mathcal{U}[f \mapsto F, x \mapsto \mathtt{w}]\mathtt{e}) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma\kappa}$, which is same as ($\star\star$).

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e_1 : (\tau_1 \xrightarrow{\mathbb{C}(\kappa')} \tau_2)^{\mu} \mid \kappa_1 \qquad \Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e_2 : \tau_1 \mid \kappa_2 \qquad \Delta; \Phi \models \mu \trianglelefteq \tau_2 \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + c_{app}(\mathbb{C}, \mu)}{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e_1\ e_2 : \tau_2 \mid \kappa}$ **app**

Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \mathcal{U}(e_1\ e_2)) \in (\!|\tau_2|\!)_\varepsilon^{\sigma\kappa}$.

Note that in the proof of this subcase, μ's value doesn't matter since $(\!|(A)^{\mu}|\!)_v = (\!|A|\!)_v$.

Assume that $\sigma(\kappa' + \kappa_1 + \kappa_2) + c_{app}(\mathbb{C}, \mu) < m$

STS:

1. $\mathcal{U}e_1\ \mathcal{U}e_2 \Downarrow \langle v_r, D\rangle, f$
2. $(m - f, v_r) \in (\!|\sigma\tau_2|\!)_v$
3. $f \leq \sigma(\kappa' + \kappa_1 + \kappa_2) + c_{app}(\mathbb{C}, \mu)$.

By IH on e_1, we get $(m, \mathcal{U}e_1) \in (\!|(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2)^{\mu}|\!)_\varepsilon^{\sigma\kappa_1}$.

Unrolling its definition using $\sigma\kappa_1 < m$, we get

a) $\mathrm{R}(\mathcal{U}e_1) \Downarrow \langle v_1, D_1\rangle, f_1$
b) $(m - f_1, \mathtt{fix}\ f(x).\, e) \in (\!|(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \sigma\tau_2)^{\mu}|\!)_v$
c) $f_1 \leq \sigma\kappa_1$

By IH on e_2, we get $(m, \mathcal{U}e_2) \in (\!|\sigma\tau_1|\!)_\varepsilon^{\sigma\kappa_2}$.

Unrolling its definition using $\sigma\kappa_2 < m$, we get

d) $\mathrm{R}(\mathcal{U}e_2) \Downarrow \langle v_2, D_2\rangle, f_2$
e) $(m - f_2, v_2) \in (\!|\sigma\tau_1|\!)_v$
f) $f_2 \leq \sigma\kappa_2$

Next, we unfold the definition of b) with $m - f_1 - f_2 - c_{app}(\mathbb{C}, \mathbb{S}) < m - f_1$ and $(m - f_2 - f_1 - c_{app}(\mathbb{C}, \mathbb{S}), v_2) \in (\!|\sigma\tau_1|\!)_v$ (obtained by applying Lemma 3 to e) with

$m - f_1 - f_2 - c_{app}(\mathbb{C}, \mathbb{S}) < m - f_2$), and we get
$(m - f_1 - f_2 - c_{app}(\mathbb{C}, \mathbb{S}), e[\texttt{fix}\ f(x).\, e/f, v/x]) \in\ (\!|\sigma\tau_2|\!)_v \sigma\kappa'$.
To unroll its definition, we first need to show that $\sigma\kappa' < m - f_1 - f_2 - c_{app}(\mathbb{C}, \mathbb{S})$.
Or, $\sigma\kappa' + f_1 + f_2 + c_{app}(\mathbb{C}, \mathbb{S}) < m$
Since $\sigma(\kappa' + \kappa_1 + \kappa_2) + c_{app}(\mathbb{C}, \mu) < m$ (by main assumption)
STS : $\sigma\kappa' + f_1 + f_2 + c_{app}(\mathbb{C}, \mathbb{S}) < \sigma(\kappa' + \kappa_1 + \kappa_2) + c_{app}(\mathbb{C}, \mu)$ This can be obtained by c) and f), $f_1 + f_2 \leq \sigma\kappa_1 + \sigma\kappa_2$. Hence, we get

g) $e[\texttt{fix}\ f(x).\, e/f, v/x] \Downarrow \langle v_r, D_r \rangle, f_r$
h) $(m - f_1 - f_2 - f_r - c_{app}(\mathbb{C}, \mathbb{S}), v_r) \in\ (\!|\sigma\tau_2|\!)_v$
i) $f_r \leq \sigma\kappa'$

We can conclude the proof by

1. By a), d) and g)
$$\frac{\begin{array}{c}\mathcal{U}e_1 \Downarrow T_1, f_1 \qquad \texttt{fix}\ f(x).\, e = \texttt{V}(T) \\ \mathcal{U}e_2 \Downarrow T_2, f_2 \qquad v_2 = \texttt{V}(T_2) \qquad e[v_2/x, (\texttt{fix}\ f(x).\, e)/f] \Downarrow T_r, f_r \qquad v_r = \texttt{V}(T_r)\end{array}}{\mathcal{U}e_1\ \mathcal{U}e_2 \Downarrow \langle v_r, \texttt{app}(T_1, T_2, T_r) \rangle, f_1 + f_2 + f_r + c_{app}(\mathbb{C}, \mathbb{S})}\ \textbf{app}$$
2. By h) and noting that $m - f = m - f_1 - f_2 - f_r - c_{app}(\mathbb{C}, \mathbb{S})$, we get $(m - f, v_r) \in\ (\!|\sigma\tau_2|\!)_v$.
3. From c), f) and i), $f_1 + f_2 + f_r + c_{app}(\mathbb{C}, \mathbb{S}) \leq \sigma(\kappa_1 + \kappa_2 + \kappa') + c_{app}(\mathbb{C}, \mu)/$
Note that $c_{app}(\mathbb{C}, \mathbb{S}) <= c_{app}(\mathbb{C}, \mu)$

Case
$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e : (\tau_1 + \tau_2)^{\mu} \mid \kappa_e \qquad \Delta; \Phi; \Gamma, x : \tau_1 \vdash_{\mathbb{C}} e_1 : \tau \mid \kappa' \\ \Delta; \Phi; \Gamma, y : \tau_2 \vdash_{\mathbb{C}} e_2 : \tau \mid \kappa' \qquad \Delta; \Phi \models \mu \trianglelefteq \tau \qquad \kappa = \kappa_e + \kappa' + c_{case}(\mathbb{C}, \mu)\end{array}}{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \texttt{case}(e, x.e_1, y.e_2) : \tau \mid \kappa}\ \textbf{case}$$
Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \mathcal{U}(\texttt{case}(e, x.e_1, y.e_2))) \in (\!|\sigma\tau|\!)^{\sigma\kappa}_{\varepsilon}$
Following the definition of $(\!|\cdot|\!)^{\cdot}_{\varepsilon}$, assume that $\sigma(\kappa_e + \kappa') + c_{case}(\mathbb{C}, \mu) < m$. STS:

1. $\mathcal{U}(\texttt{case}(e, x.e_1, y.e_2)) \Downarrow \langle v_r, D'_r \rangle, f$
2. $(m - f, v_r) \in\ (\!|\sigma\tau|\!)_v$
3. $f \leq \sigma\kappa$

By IH on e, $(m, \mathcal{U}e) \in (\!|(\sigma\tau_1 + \sigma\tau_2)^{\mu}|\!)^{\sigma\kappa_e}_{\varepsilon}$.
Unrolling this using $\sigma\kappa_e < m$, we get

a) $\mathcal{U}e \Downarrow \langle v_e, D_e \rangle, f_e$
b) $(m - f_e, v_e) \in (\!|(\sigma\tau_1 + \sigma\tau_2)^{\mu}|\!)_v$
c) $f_e < \sigma\kappa_e$

There are two cases for a). We only show the first case, $v_e = \texttt{inl}\ v$, as the other one is similar. Then, by c), we get $(m - f_e, v) \in (\!|\sigma\tau_1|\!)_v(\star)$. By IH on e_1 using $(m - f_e, \mathcal{U}[x \mapsto v]) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau_1|\!)$, which holds because

* $(m - f_e, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by Lemma 3 using $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $m - f_e \leq m$,
* by $(\star)$

hence we get $(m - f_e, \mathcal{U}[x \mapsto v]e_1) \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma\kappa'}$.
To unroll its definition we need to show $\sigma\kappa' < m - f_e$
We know that by b) $f_e \leq \sigma\kappa'$
hence we get $\sigma\kappa_e + f_e \leq \sigma\kappa' + \sigma\kappa_e \leq \sigma\kappa_e + \sigma\kappa_e + c_{case}(\mathbb{C}, \mathbb{S}) < m$. Hence we can unroll and get

d) $\mathcal{U}[x \mapsto v]e_1 \Downarrow \langle v_r, D_r \rangle, f_e$
e) $(m - f_e - f_r, v_r) \in (\!|\sigma\tau|\!)_v$
f) $f_r < \sigma\kappa'$

Then, we conclude by showing

1. By a) and d)
$$\frac{\mathcal{U}e \Downarrow T_e, f_e \qquad \texttt{inl}\ v = \texttt{V}(T_r) \qquad \mathcal{U}e_1[v/x] \Downarrow T_r, f_r \qquad v_r = \texttt{V}(T_r)}{\mathcal{U}(\texttt{case}(e, x.e_1, y.e_2)) \Downarrow \langle v_r, \texttt{case}_{\texttt{inl}}(T_e, T_r) \rangle, f_e + f_r + c_{case}(\mathbb{C}, \mathbb{S})}\ \textbf{r-case-inl}$$
2. By Lemma 3 using e) and $m - f = m - f_e - f_r - c_{case}(\mathbb{C}, \mathbb{S}) \leq m - f_e - f_r$, we get $(m - f, v_r) \in (\!|\sigma\tau|\!)_v$
3. c) and f) we get $f_e + f_r + c_{case}(\mathbb{C}, \mathbb{S}) \leq \sigma\kappa_e + \sigma\kappa' + c_{case}(\mathbb{C}, \mu)$

Case
$$\frac{\begin{array}{c}\Upsilon(\zeta) = \zeta : (B_1 \cdots B_n) \xrightarrow{\kappa'} B \qquad \Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e : (B_i)^{\mu_i} \mid \kappa_{e_i} \\ \mu_1 \sqcup \cdots \sqcup \mu_n = \mu \qquad \kappa = (\sum_{i=1}^{n} \kappa_{e_i}) + \kappa' + c_{prim}(\mathbb{C}, n, \mu_1, \cdots, \mu_n)\end{array}}{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \zeta\ (e_1 \cdots e_n) : (B)^{\mu} \mid \kappa}\ \textbf{primApp}$$

Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \mathcal{U}(\zeta\ (e_1 \cdots e_n))) \in (\!|B|\!)_\varepsilon^{\sigma\kappa}$
Following the definition of $(\!|\cdot|\!)_\varepsilon$, assume that $(\sum_{i=1}^{n} \sigma\kappa_{e_i}) + \kappa' + c_{prim}(\mathbb{C}, n, \mu_1, \cdots, \mu_n) < m$.
STS:

1. $\mathcal{U}(\zeta\ (e_1 \cdots e_n)) \Downarrow \langle v, D \rangle, f$
2. $(m - f, v) \in (\!|\sigma\tau|\!)_v$
3. $f \leq \sigma\kappa$

By IH on each premise e_i we get $(m, \mathcal{U}e_i) \in (\!|(B_i)^{\mu_i}|\!)_\varepsilon^{\sigma\kappa_{e_i}}$.
By unrolling each with $\sigma\kappa_{e_i} < m$, we get

a) $\mathcal{U}e_i \Downarrow \langle v_i, D_i\rangle, f_i$

b) $(m - f_i, v_i) \in \ (\!|(B_i)^{\mu_i}|\!)_v$

c) $f_i \leq \sigma\kappa_{e_i}$

Then using Assumption 18, we get

d) $\widehat{\zeta}(v_1 \cdots v_n) = (f_r, v_r)$

e) $(m, v_r) \in (\!|(B)^{\mu}|\!)_v$

f) $f_r \leq \kappa'$

Now we can conclude

1. Using a) and d)
$$\frac{\mathcal{U}e_i \Downarrow T_i, f_i \qquad v_i = \mathtt{V}(T_i) \qquad \widehat{\zeta}(v_1 \cdots v_n) = (f_r, v_r)}{\zeta\ (\mathcal{U}e_1 \cdots \mathcal{U}e_n) \Downarrow \langle v_r, \mathtt{primApp}(T_1 \cdots T_n, \zeta)\rangle, (\sum_{i=1}^{n} f_i) + f_r + c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S})}\ \textbf{primapp}$$
2. By Lemma 3 using e) and $m - f = m - (\sum_{i=1}^{n} f_i) - f_r - c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S}) \leq m$, we get $(m - f, v_r) \in \ (\!|B|\!)_v$
3. By c) and f) $(\sum_{i=1}^{n} f_i) + f_r + c_{prim}(\mathbb{C}, n, \mathbb{S}, \cdots, \mathbb{S}) \leq (\sum_{i=1}^{n} \sigma\kappa_{e_i}) + \kappa' + c_{prim}(\mathbb{C}, n, \mu_1, \cdots, \mu_n)$

Proof of statement (3):

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.

Case $\dfrac{\Delta; \Phi; f : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}, x : \tau_1, \Gamma \vdash_{\delta} e : \tau_2 \mid \kappa}{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \mathtt{fix}\ f(x).\, e : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}} \mid c_{fix}()}\ \textbf{fix1}$

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta^{\ulcorner}\mathtt{fix}\ f(x).\, e^{\urcorner}) \in (\!|(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)_{\varepsilon}^{c_{fix}()}$.

By unfolding the definition using `fix` evaluation rule which takes $c_{fix}()$ step,

STS: $(m - c_{fix}(), \theta^{\ulcorner}\mathtt{fix}\ f(x).\, e^{\urcorner}) \in \ [\![(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$. Let $F = \theta^{\ulcorner}\mathtt{fix}\ f(x).\, e^{\urcorner}$.

There are two cases.

subcase 1: $\delta = \mathbb{S}$

We prove the more general statement

$\forall k \leq m - c_{fix}().\ (k, \theta^{\ulcorner}\mathtt{fix}\ f(x).\, e^{\urcorner}) \in \ [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by subinduction on k.

subsubcase 1: $k = 0$ is vacuous from the definition $[\![\cdot]\!]_v$ at the function type.

subsubcase 2: $k + 1 \leq m - c_{fix}()$

Assume, by the sub-IH, that $(k, F) \in \ [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ ($\star$)

STS: $(k + 1, F) \in \ [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$

Following the definition, pick $j < k + 1$.

Assume that $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$. Then, STS: $(j, \theta^{\ulcorner} e^{\urcorner}[F/f, \mathtt{w}/\mathtt{x}]) \in [\![\sigma\tau_2]\!]_{\varepsilon}^{\sigma\kappa}$ ($\star\star$).

Instantiate the IH 1 on the premise of the typing rule using:

$(j, \theta[f \mapsto F, x \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma(\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : (\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \tau_2)^{\mathbb{S}})]\!]$, which holds because:

* $(j, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ and $j \leq m$,
* $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$
* $(j, F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on ($\star$) and $j \leq k$

We immediately get $(j, \theta[f \mapsto F, x \mapsto \mathtt{w}]^{\ulcorner} \mathtt{e}^{\urcorner}) \in [\![\sigma\tau_2]\!]_{\varepsilon}^{\sigma\kappa}$, which is same as ($\star\star$).

subcase 2: $\delta = \mathbb{C}$

Proof of this case is very similar to the one in the $\mathbb{S}$ typing judgment.

□

Theorem 20 (Fundamental theorem for bi-values and bi-expressions)
The following hold.

1. If $\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{w} \gg \tau$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$, then $(m, \theta(\mathtt{w})) \in [\![\sigma\tau]\!]_{\mathtt{v}}$
2. If $\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{w} \gg \tau$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(k, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$, then $(k, \mathcal{U}(\mathrm{L}(\mathtt{w}))) \in (\!|\sigma\tau|\!)_{\mathtt{v}}$ and $(k, \mathcal{U}(\mathrm{R}(\mathtt{w}))) \in (\!|\sigma\tau|\!)_{\mathtt{v}}$.
3. If $\Delta; \Phi; \Gamma \vdash_{\mathbb{S}}^{\kappa} \mathtt{ee} \gg \tau$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$, then $(m, \theta(\mathtt{ee})) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa}$.
4. If $\Delta; \Phi; \Gamma \vdash_{\mathbb{C}}^{\kappa} \mathtt{ee} \gg \tau$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(k, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$, then $(k, \mathcal{U}(\mathrm{L}(\mathtt{ee}))) \in (\!|\sigma\tau|\!)_{\varepsilon}^{\sigma\kappa}$ and $(k, \mathcal{U}(\mathrm{R}(\mathtt{ee}))) \in (\!|\sigma\tau|\!)_{\varepsilon}^{\sigma\kappa}$.
5. If $\Delta; \Phi; \Gamma \vdash_{\mathbb{C}}^{\kappa} \mathtt{ee} \gg \tau$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$, then $(m, \theta(\mathtt{ee})) \in [\![\sigma\tau]\!]_{\varepsilon}^{\sigma\kappa}$.

Proof. All the statements are proved by simultaneous induction on bi-value and bi-expression typing. We show select cases of the proofs. In these proofs, the numbers 1–5 represent the corresponding clauses in the definition of $[\![\tau]\!]_{\varepsilon}^{\kappa}$.

Proof of statement (1):

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$.

Case $\dfrac{}{\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{keep}(\mathtt{r}) \gg (\mathtt{real})^{\mathbb{S}}}$ **keep-r**

TS: $(m, \theta\ \mathtt{keep}(\mathtt{r})) \in [\![(\mathtt{real})^{\mathbb{S}}]\!]_v$.

This follows from the definition of $[\![(\mathtt{real})^{\mathbb{S}}]\!]_v$.

Case $\dfrac{\Delta; \Phi; \cdot \vdash_{\mathbb{C}} \mathtt{v} : \tau \mid \kappa \qquad \Delta; \Phi; \cdot \vdash_{\mathbb{C}} \mathtt{v}' : \tau \mid \kappa' \qquad \models \mathbb{C} \trianglelefteq \tau}{\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{new}(v, v') \gg \tau}$ **new**

TS: $(m, \theta\ \mathtt{new}(v, v')) \in [\![\sigma\tau]\!]_v$.

Since $\models \mathbb{C} \trianglelefteq \tau$ and $\mathtt{v}, \mathtt{v}'$ are values, STS: $\forall k.(k, \mathtt{v}) \in (\!|\sigma\tau|\!)_v(\star) \ \wedge \ (k, \mathtt{v}') \in (\!|\sigma\tau|\!)_v(\star\star)$.

Pick k as $\mathtt{max}(\sigma\kappa, \sigma\kappa') + 1$.

We know that $(\mathtt{max}(\sigma\kappa, \sigma\kappa') + 1, \emptyset) \in \mathcal{G}(\!|\cdot|\!)$.

By the (Fundamental) Theorem 19 (2^{nd} clause) on the first premise, we get
$(\mathtt{max}(\sigma\kappa, \sigma\kappa') + 1, \mathtt{v}) \in (\!|\sigma\tau|\!)^{\kappa}_{\varepsilon}$.
By unfolding its definition using $\sigma\kappa < \mathtt{max}(\sigma\kappa, \sigma\kappa') + 1$, we get

a) $\mathtt{v} \Downarrow \langle \mathtt{v}, \mathtt{v} \rangle, 0$
b) $(\mathtt{max}(\sigma\kappa, \sigma\kappa') + 1, \mathtt{v}) \in (\!|\sigma\tau|\!)_v$

Similarly, by Theorem 19 (2^{nd} clause) on the second premise, we get
$(\mathtt{max}(\kappa, \kappa') + 1, \mathtt{v}') \in (\!|\sigma\tau|\!)^{\kappa}_{\varepsilon}$.
By unfolding its definition using $\sigma\kappa' < \mathtt{max}(\sigma\kappa, \sigma\kappa') + 1$, we get

c) $\mathtt{v}' \Downarrow \langle \mathtt{v}', \mathtt{v}' \rangle, 0$
d) $(\mathtt{max}(\sigma\kappa, \sigma\kappa') + 1, \mathtt{v}') \in (\!|\sigma\tau|\!)_v$

b) and d) complete the proof.

Case $\dfrac{\Delta; \Phi; x : \tau_1, f : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}, \Gamma \vdash^{\kappa}_{\delta} \mathtt{ee} \gg \tau_2}{\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{fix}\ f(x).\, \mathtt{ee} \gg (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}}$ **fix1**

Assume that $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.
TS: $(m, \theta(\mathtt{fix}\ f(x).\, \mathtt{ee})) \in [\![(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_{\mathtt{v}}$.
Let $F = \theta(\mathtt{fix}\ f(x).\, \mathtt{ee})$.
There are two cases.

subcase 1: $\delta = \mathbb{S}$

We prove the more general statement
$\forall k \leq m.\ (k, \mathtt{fix}\ f(x).\, \theta(\mathtt{ee})) \in [\![(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_{\mathtt{v}}$ by subinduction on k.

subsubcase 1: $k = 0$ is vacuous from the definition $[\![\cdot]\!]_v$ at the function type.

subsubcase 2: $k + 1 \leq m$

Assume, by the sub-IH, that $(k, F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ ($\star$)
STS: $(k+1, F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$
Following the definition, pick $j < k + 1$.
Assume that $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$. Then, STS: $(j, \theta(\mathtt{ee})[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}]) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$ ($\star\star$).
Instantiate the IH 2 on the premise of the typing rule using $\delta = \mathbb{S}$ and
$(j, \theta[f \mapsto F, x \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma(\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}})]\!]$, which holds because:

* $(j, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by Lemma 3 using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ and $j \leq m$,
* $(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$
* $(j, F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on ($\star$) and $j \leq k$

We immediately get $(j, \theta[f \mapsto F, x \mapsto \mathtt{w}]\mathtt{ee}) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$, which is same as ($\star\star$).

subcase 2: $\sigma\delta = \mathbb{C}$

There are two cases.

- **STS:** $\forall v, j < m.(j, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta\mathtt{ee}[\mathtt{F}/\mathtt{f}, \mathtt{w}/\mathtt{x}]) \in (\!|\sigma\tau_2|\!)^{\sigma\kappa}_{\varepsilon}$

  Proof by sub-induction on m.

  i. **case** $m = 0$ is vacuous since there exists no positive j < 0.

  ii. **case** $m = m' + 1$

  STS: $\forall j < m' + 1.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta\mathtt{ee}[\mathtt{F}/\mathtt{f}][\mathtt{w}/\mathtt{x}]) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$.

  There are two possible cases.

  – $j < m'$

  Then, by sub-IH, we know that $\forall j < m'.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta\mathtt{ee}[\mathtt{F}/\mathtt{f}][\mathtt{w}/\mathtt{x}]) \in [\![\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$. Since $j < m' < m' + 1$, we can immediately conclude.

  – $j = m'$

  Since $j = m' < m' + 1$, we assume that $(m', \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}(\diamond)$.

  STS: $(m', \theta\mathtt{ee}[\mathtt{F}/\mathtt{f}][\mathtt{w}/\mathtt{x}]) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}(\diamond\diamond)$

  By IH 5 on the premise of the typing rule using $(m', \theta\mathtt{ee}[\mathtt{f} \mapsto \mathtt{F}, \mathtt{x} \mapsto \mathtt{w}]) \in \mathcal{G}[\![\sigma\Gamma, \mathtt{x} : \sigma\tau_1, \mathtt{f} : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]$, which holds because:

  $*$ $(m', \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ by instantiating Lemma 3 with $m' < m$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$

  $*$ $(m', \mathtt{w}) \in [\![\sigma\tau_1]\!]_{\mathtt{v}}$ by $(\diamond)$

  $*$ $(m', F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ since as shown in (STS 1) above, $\forall k.\ (k, F) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$ and by sub-IH, $\forall j < m'.\ \forall \mathtt{w}.\ (\mathtt{j}, \mathtt{w}) \in [\![\tau_1]\!]_{\mathtt{v}} \Rightarrow (\mathtt{j}, \theta\mathtt{ee}[\mathtt{F}/\mathtt{f}][\mathtt{w}/\mathtt{x}]) \in [\![\tau_2]\!]^{\kappa}_{\varepsilon}$.

  We immediately get $(m', \theta[f \mapsto F, x \mapsto \mathtt{w}]\mathtt{ee}) \in [\![\sigma\tau_2]\!]^{\sigma\kappa}_{\varepsilon}$, which is same as $(\diamond\diamond)$.

- **STS :** $\forall k.\ (k, \mathrm{L}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$ and $(k, \mathrm{R}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$.

  Proof proceeds by sub-induction on k.

  i. **case** $k = 0$ is vacuous from the definition of $[\![\cdot]\!]_v$ at the function type with C body.

  ii. **case** $k + 1$

  Assume by sub-IH that $(k, \mathrm{L}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v(\dagger)\ \wedge\ (k, \mathrm{R}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v(\dagger\dagger)$.

  STS: $(k+1, \mathrm{L}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v\ \wedge\ (k+1, \mathrm{R}(F)) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$

  Pick $j < k + 1$ s.t. $(j, v) \in (\!|\sigma\tau_1|\!)_v$. Then,

  STS1: $(j, \mathrm{L}(\theta\mathtt{ee})[\mathrm{L}(\mathtt{F})/\mathtt{f}, \mathtt{v}/\mathtt{x}]) \in (\!|\tau_2|\!)^{\sigma\kappa}_{\varepsilon}(\diamond)$

  STS2: $(j, \mathrm{R}(\theta\mathtt{ee})[\mathrm{R}(\mathtt{F})/\mathtt{f}, \mathtt{v}/\mathtt{x}]) \in (\!|\tau_2|\!)^{\sigma\kappa}_{\varepsilon}(\diamond\diamond)$

  We first show the first one.

  Instantiate the IH 4 on the premise of the typing rule using $\delta = \mathbb{C}$ and $(j, \mathrm{L}(\theta)\mathtt{ee}[\mathtt{f} \mapsto \mathrm{L}(\mathtt{F}), \mathtt{x} \mapsto \mathtt{v}]) \in \mathcal{G}(\!|\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)$, which holds because:

  $*$ $(j, \mathrm{L}(\theta)) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by instantiating Lemma 2 with j using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$,

  $*$ $(j, v) \in (\!|\sigma\tau_1|\!)_v$

$*$ $(j, \mathrm{L}(F)) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on (†) using $j \leq k$

We immediately get $(j, \mathrm{L}(\theta)[f \mapsto \mathrm{L}(F), x \mapsto v]\mathrm{R}(\text{ee})) \in (\!|\sigma\tau_2|\!)^{\sigma\kappa}_{\varepsilon}$, which is same as (⋄).

Instantiate the IH 4 on the premise of the typing rule using $\delta = \mathbb{C}$ and $(j, \mathrm{R}(\theta)\text{ee}[\mathtt{f} \mapsto \mathrm{R}(\mathtt{F}), \mathtt{x} \mapsto \mathtt{v}]) \in \mathcal{G}(\!|\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)$, which holds because:

$*$ $(j, \mathrm{R}(\theta)) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by instantiating Lemma 2 with j using $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$,

$*$ $(j, v) \in (\!|\sigma\tau_1|\!)_v$

$*$ $(j, \mathrm{R}(F)) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on (††) using $j \leq k$

We immediately get $(j, \mathrm{R}(\theta)[f \mapsto \mathrm{R}(F), x \mapsto v]\mathrm{R}(\text{ee})) \in (\!|\sigma\tau_2|\!)^{\sigma\kappa}_{\varepsilon}$, which is same as (⋄⋄).

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{w} \gg \tau \qquad \forall z \in \Gamma.\ \ \Delta; \Phi \models \Gamma(z) \sqsubseteq \square(\Gamma(z)) \qquad \mathtt{stable}(\mathtt{w})}{\Delta; \Phi; \Gamma, \Gamma' \vdash_{\epsilon} \mathtt{w} \gg \square(\tau)}$ **nochange**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma, \sigma\Gamma']\!]$ and $\models \sigma\Phi$.

Let $\theta = \theta_1 \cup \theta_2$ where $(m, \theta_1) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $(m, \theta_2) \in \mathcal{G}[\![\sigma\Gamma']\!]$.

TS: $(m, \theta(\mathtt{w})) \in [\![\square(\sigma\tau)]\!]_{\mathtt{v}}$.

STS: $(m, \theta_1(\mathtt{w})) \in [\![\sigma\tau]\!]_{\mathtt{v}}$ and $\mathtt{stable}(\theta_1(\mathtt{w}))$, since $\mathtt{w}$ doesn't have any free variables from Γ'.

By Lemma 7 on $\mathtt{stable}(\mathtt{w})$ and $\forall z \in \Gamma.\ \Delta; \Phi \models \Gamma(z) \sqsubseteq \square(\Gamma(z))$, we get $\mathtt{stable}(\theta_1(\mathtt{w}))$.

By IH on $\mathtt{w}$, we get $(m, \theta_1(\mathtt{w})) \in [\![\sigma\tau]\!]_{\mathtt{v}}$.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{w} \gg \tau \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{w} \gg \tau'}\ \sqsubseteq$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta\mathtt{w}) \in [\![\sigma\tau']\!]_{\mathtt{v}}$.

By IH on the premise, $(m, \theta(\mathtt{w})) \in [\![\sigma\tau]\!]_{\mathtt{v}}$.

By Lemma 11 $(m, \theta(\mathtt{w})) \in [\![\sigma\tau']\!]_{\mathtt{v}}$.

Proof of statement (2):

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.

Case $\dfrac{}{\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{keep}(\mathtt{r}) \gg (\mathtt{real})^{\mathbb{S}}}$ **keep-r**

Assume that $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS1: $(m, \mathcal{U}\ \mathrm{L}(\mathtt{keep}(\mathtt{r}))) \in (\!|(\mathtt{real})^{\mathbb{S}}|\!)_v$.

TS2: $(m, \mathcal{U}\ \mathrm{R}(\mathtt{keep}(\mathtt{r}))) \in (\!|(\mathtt{real})^{\mathbb{S}}|\!)_v$.

These follow from the definition of $(\!|(\mathtt{real})^{\mathbb{S}}|\!)_v$.

Case $\dfrac{\Delta; \Phi; \cdot \vdash_{\mathbb{C}} \mathtt{v} : \tau \mid \kappa \qquad \Delta; \Phi; \cdot \vdash_{\mathbb{C}} \mathtt{v}' : \tau \mid \kappa' \qquad \Delta; \Phi \models \mathbb{C} \trianglelefteq \tau}{\Delta; \Phi; \Gamma \vdash_{\epsilon} \mathtt{new}(v, v') \gg \tau}$ **new**

Assume that $(m,\mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m,\mathcal{U}\ \mathrm{L}(\mathtt{new}(v,v'))) \in\ (\!|\sigma\tau|\!)_v$ and $(m,\mathcal{U}\ \mathrm{R}(\mathtt{new}(v,v'))) \in\ (\!|\sigma\tau|\!)_v$.

STS1: $(m,\mathtt{v}) \in\ (\!|\sigma\tau|\!)_v(\star)$.

STS2: $(m,\mathtt{v}') \in\ (\!|\sigma\tau|\!)_v(\diamond)$.

By (Fundamental) Theorem 19 on the first and second premises with
$(m+\mathtt{max}(\sigma\kappa,\sigma\kappa')+1,\emptyset) \in \mathcal{G}(\!|\cdot|\!)$, we get
$(m+\mathtt{max}(\sigma\kappa,\sigma\kappa')+1,v) \in\ (\!|\sigma\tau|\!)^{\sigma\kappa}_{\varepsilon}$ and $(m+\mathtt{max}(\sigma\kappa,\sigma\kappa')+1,v') \in\ (\!|\sigma\tau|\!)^{\sigma\kappa'}_{\varepsilon}$.
Since $\sigma\kappa < m+\mathtt{max}(\sigma\kappa,\sigma\kappa')+1$, we get

a) $v \Downarrow \langle v,v\rangle, 0$

b) $(m+\mathtt{max}(\sigma\kappa,\sigma\kappa')+1-0,v) \in\ (\!|\sigma\tau|\!)_v$

c) $0 \le \sigma\kappa$

Since $\sigma\kappa' < m+\mathtt{max}(\sigma\kappa,\sigma\kappa')+1$.

d) $v' \Downarrow \langle v',v'\rangle, 0$

e) $(m+\mathtt{max}(\sigma\kappa,\sigma\kappa')+1-0,v') \in\ (\!|\sigma\tau|\!)_v$

f) $0 \le \sigma\kappa'$

$(\star)$ and $(\diamond)$ follow by Lemma 3 on b) and e) with $m \le m+\mathtt{max}(\sigma\kappa,\sigma\kappa')+1$.

Case $\dfrac{\Delta;\Phi;x:\tau_1,f:(\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}},\Gamma \vdash^{\kappa}_{\delta} \mathtt{ee} \gg \tau_2}{\Delta;\Phi;\Gamma \vdash_{\epsilon} \mathtt{fix}\ f(x).\,\mathtt{ee} \gg (\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mathbb{S}}}$ **fix1**

Assume that $(m,\mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m,\mathcal{U}(\mathrm{R}(\mathtt{fix}\ f(x).\,\mathtt{ee}))) \in\ (\!|(\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)_{\mathtt{v}}$.

STS: $(m,\mathcal{U}(\mathrm{R}(\mathtt{fix}\ f(x).\,\mathtt{ee}))) \in\ (\!|\sigma\tau_1 \xrightarrow{\delta(\sigma\kappa)} \sigma\tau_2|\!)_{\mathtt{v}}$.

Let $F = \mathcal{U}(\mathrm{R}(\mathtt{fix}\ f(x).\,\mathtt{ee}))$.

There are two cases.

subcase 1: $\delta = \mathbb{S}$

By definition, any function is in $(\!|\sigma\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \sigma\tau_2|\!)_v$

subcase 2: $\sigma\delta = \mathbb{C}$

We show the more general statement $\forall k.\ (k,F) \in\ (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$.

Proof proceeds by sub-induction on k.

i. **case** $k=0$ is vacuous from the definition of $[\![\cdot]\!]_v$ at the function type with C body.

ii. **case** $k+1$

Assume by sub-IH that $(k,F) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v(\dagger)$.

STS: $(k+1,F) \in (\!|\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2|\!)_v$

Pick $j < k+1$ s.t. $(j,v) \in\ (\!|\sigma\tau_1|\!)_v$. Then, STS: $(j,\mathcal{U}(\mathrm{R}(\mathtt{ee}))[\mathtt{F}/\mathtt{f},\mathtt{v}/\mathtt{x}]) \in\ (\!|\tau_2|\!)^{\sigma\kappa}_{\varepsilon}(\dagger\dagger)$

Instantiate the IH 4 on the premise of the typing rule using $\delta = \mathbb{C}$ and $(j, \mathcal{U}(\mathrm{R}(\text{ee}))[\mathtt{f} \mapsto \mathtt{F}, \mathtt{x} \mapsto \mathtt{v}]) \in \mathcal{G}(\!|\Gamma, \mathtt{x} : \tau_1, \mathtt{f} : (\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}|\!)$, which holds because:

* $(j, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by Lemma 3 on $(m, \theta) \in \mathcal{G}[\![\Gamma]\!]$ using $j \leq k$,
* $(j, v) \in (\!|\sigma\tau_1|\!)_v$
* $(j, F) \in [\![(\sigma\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa)} \sigma\tau_2)^{\mathbb{S}}]\!]_v$ by Lemma 3 on (†) using $j \leq k$

We immediately get $(j, \mathcal{U}[f \mapsto F, x \mapsto v]\mathrm{R}(\text{ee})) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma\kappa}$, which is same as (††).

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{w} \gg \tau \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{w} \gg \tau'}\ \sqsubseteq$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \mathcal{U}(\mathrm{R}(\mathtt{w}))) \in (\!|\sigma\tau'|\!)_{\mathtt{v}}$.

By IH 2 on the premise, $(m, \mathcal{U}(\mathrm{R}(\mathtt{w}))) \in (\!|\sigma\tau|\!)_{\mathtt{v}}$.

By Lemma 11 $(m, \mathcal{U}(\mathrm{R}(\mathtt{w}))) \in (\!|\sigma\tau'|\!)_{\mathtt{v}}$.

Proof of statement 3:

There is only one case:

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \mathtt{w_i} \gg \tau_{\mathtt{i}} \qquad \Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa}{\Delta; \Phi; \Gamma \vdash^{\kappa}_{\mathbb{S}} \ulcorner e \urcorner[\overline{\mathtt{w_i}/\mathtt{x_i}}] \gg \tau}$ **exp**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta\ulcorner e \urcorner[\overline{\theta(\mathtt{w_i})/\mathtt{x_i}}]) \in [\![\sigma\tau]\!]_\varepsilon^{\sigma\kappa}$ $(*)$.

By IH(1) on premise $\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} \mathtt{w_i} \gg \tau_{\mathtt{i}}$, we get $(m, \theta(\mathtt{w_i})) \in [\![\sigma\tau_{\mathtt{i}}]\!]_{\mathtt{v}}$.

By (Fundamental) Theorem 19. on premise $\Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa$ using

$\sigma \in \mathcal{D}[\![\Delta]\!]$,

$(m, \theta[x_i \mapsto \theta(\mathtt{w_i})]) \in \mathcal{G}[\![\mathtt{x_i} : \sigma\tau_{\mathtt{i}}, \sigma\Gamma]\!]$ (since $(m, \theta(\mathtt{w_i})) \in [\![\sigma\tau_{\mathtt{i}}]\!]_{\mathtt{v}}$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$) and

$\models \sigma\Phi$,

we get $(m, \theta[x_i \mapsto \theta(\mathtt{w_i})]\text{ee}) \in [\![\sigma\tau]\!]_\varepsilon^{\sigma\kappa}$ which is the same as $(*)$.

Proof of statement 4:

There is only one case:

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \mathtt{w_i} \gg \tau_{\mathtt{i}} \qquad \Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_{\mathbb{C}} e : \tau \mid \kappa}{\Delta; \Phi; \Gamma \vdash^{\kappa}_{\mathbb{C}} \ulcorner e \urcorner[\overline{\mathtt{w_i}/\mathtt{x_i}}] \gg \tau}$ **exp**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS1: $(m, \mathcal{U}\mathrm{L}(\ulcorner e \urcorner)[\overline{\mathcal{U}(\mathrm{L}(\mathtt{w_i}))/\mathtt{x_i}}]) \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma\kappa}$ $(\star)$.

TS2: $(m, \mathcal{U}\mathrm{R}(\ulcorner e \urcorner)[\overline{\mathcal{U}(\mathrm{R}(\mathtt{w_i}))/\mathtt{x_i}}]) \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma\kappa}$ $(\star\star)$.

We first show the first statement.

By IH(1) on premise $\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \mathtt{w_i} \gg \tau_{\mathtt{i}}$, we get $(m, \mathcal{U}(\mathrm{L}(\mathtt{w_i}))) \in (\!|\sigma\tau_{\mathtt{i}}|\!)_{\mathtt{v}}$.

By (Fundamental) Theorem 19. on premise $\Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_{\mathbb{C}} e : \tau \mid \kappa$ using

$\sigma \in \mathcal{D}[\![\Delta]\!]$,

$(m, \theta[x_i \mapsto \mathcal{U}(\mathrm{L}(\mathtt{w_i}))]) \in \mathcal{G}(\!|\mathtt{x_i} : \sigma\tau_\mathtt{i}, \sigma\Gamma|\!)$ (since $(m, \mathcal{U}(\mathrm{L}(\mathtt{w_i}))) \in (\!|\sigma\tau_\mathtt{i}|\!)_\mathtt{v}$ and $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$) and

$\models \sigma\Phi$,

we get $(m, \mathcal{U}[x_i \mapsto \mathcal{U}(\mathrm{L}(\mathtt{w_i}))]\mathtt{e}) \in (\!|\sigma\tau|\!)^{\sigma\kappa}_\varepsilon$ which is the same as ($\star$).

By IH(1) on premise $\Delta; \Phi; \Gamma \vdash_\mathbb{C} \mathtt{w_i} \gg \tau_\mathtt{i}$, we get $(m, \mathcal{U}(\mathrm{R}(\mathtt{w_i}))) \in (\!|\sigma\tau_\mathtt{i}|\!)_\mathtt{v}$.
By (Fundamental) Theorem 19. on premise $\Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_\mathbb{C} e : \tau \mid \kappa$ using

$\sigma \in \mathcal{D}[\![\Delta]\!]$,

$(m, \theta[x_i \mapsto \mathcal{U}(\mathrm{R}(\mathtt{w_i}))]) \in \mathcal{G}(\!|\mathtt{x_i} : \sigma\tau_\mathtt{i}, \sigma\Gamma|\!)$ (since $(m, \mathcal{U}(\mathrm{R}(\mathtt{w_i}))) \in (\!|\sigma\tau_\mathtt{i}|\!)_\mathtt{v}$ and $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$) and

$\models \sigma\Phi$,

we get $(m, \mathcal{U}[x_i \mapsto \mathcal{U}(\mathrm{R}(\mathtt{w_i}))]\mathtt{e}) \in (\!|\sigma\tau|\!)^{\sigma\kappa}_\varepsilon$ which is the same as ($\star\star$).

Proof of statement 4:

There is only one case:

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_\mathbb{C} \mathtt{w_i} \gg \tau_\mathtt{i} \qquad \Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_\mathbb{C} e : \tau \mid \kappa}{\Delta; \Phi; \Gamma \vdash^\kappa_\mathbb{C} \ulcorner e \urcorner[\overline{\mathtt{w_i}/\mathtt{x_i}}] \gg \tau}$ **exp**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS1: $(m, \mathcal{U}\mathrm{L}(\ulcorner e \urcorner)[\overline{\mathcal{U}(\mathrm{L}(\mathtt{w_i}))/\mathtt{x_i}}]) \in (\!|\sigma\tau|\!)^{\sigma\kappa}_\varepsilon$ ($\star$).
TS2: $(m, \mathcal{U}\mathrm{R}(\ulcorner e \urcorner)[\overline{\mathcal{U}(\mathrm{R}(\mathtt{w_i}))/\mathtt{x_i}}]) \in (\!|\sigma\tau|\!)^{\sigma\kappa}_\varepsilon$ ($\star\star$).
We first show the first statement.
By IH(1) on premise $\Delta; \Phi; \Gamma \vdash_\mathbb{C} \mathtt{w_i} \gg \tau_\mathtt{i}$, we get $(m, \mathcal{U}(\mathrm{L}(\mathtt{w_i}))) \in (\!|\sigma\tau_\mathtt{i}|\!)_\mathtt{v}$.
By (Fundamental) Theorem 19. on premise $\Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_\mathbb{C} e : \tau \mid \kappa$ using

$\sigma \in \mathcal{D}[\![\Delta]\!]$,

$(m, \theta[x_i \mapsto \mathcal{U}(\mathrm{L}(\mathtt{w_i}))]) \in \mathcal{G}(\!|\mathtt{x_i} : \sigma\tau_\mathtt{i}, \sigma\Gamma|\!)$ (since $(m, \mathcal{U}(\mathrm{L}(\mathtt{w_i}))) \in (\!|\sigma\tau_\mathtt{i}|\!)_\mathtt{v}$ and $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$) and

$\models \sigma\Phi$,

we get $(m, \mathcal{U}[x_i \mapsto \mathcal{U}(\mathrm{L}(\mathtt{w_i}))]\mathtt{e}) \in (\!|\sigma\tau|\!)^{\sigma\kappa}_\varepsilon$ which is the same as ($\star$).

By IH(1) on premise $\Delta; \Phi; \Gamma \vdash_\mathbb{C} \mathtt{w_i} \gg \tau_\mathtt{i}$, we get $(m, \mathcal{U}(\mathrm{R}(\mathtt{w_i}))) \in (\!|\sigma\tau_\mathtt{i}|\!)_\mathtt{v}$.
By (Fundamental) Theorem 19. on premise $\Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_\mathbb{C} e : \tau \mid \kappa$ using

$\sigma \in \mathcal{D}[\![\Delta]\!]$,

$(m, \theta[x_i \mapsto \mathcal{U}(\mathrm{R}(\mathtt{w_i}))]) \in \mathcal{G}(\!|\mathtt{x_i} : \sigma\tau_\mathtt{i}, \sigma\Gamma|\!)$ (since $(m, \mathcal{U}(\mathrm{R}(\mathtt{w_i}))) \in (\!|\sigma\tau_\mathtt{i}|\!)_\mathtt{v}$ and $(m, \mathcal{U}) \in \mathcal{G}(\!|\sigma\Gamma|\!)$) and

$\models \sigma\Phi$,

we get $(m, \mathcal{U}[x_i \mapsto \mathcal{U}(\mathrm{R}(\mathtt{w_i}))]\mathtt{e}) \in (\!|\sigma\tau|\!)^{\sigma\kappa}_{\varepsilon}$ which is the same as ($\star\star$).

Proof of statement 5:

There is only one case:

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \mathtt{w_i} \gg \tau_{\mathtt{i}} \qquad \Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_{\mathbb{C}} e : \tau \mid \kappa}{\Delta; \Phi; \Gamma \vdash^{\kappa}_{\mathbb{C}} \ulcorner e \urcorner[\overline{\mathtt{w_i}/\mathtt{x_i}}] \gg \tau}$ **exp**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$ and $\models \sigma\Phi$.

TS: $(m, \theta\ulcorner e \urcorner[\overline{\theta(\mathtt{w_i})/\mathtt{x_i}}]) \in [\![\sigma\tau]\!]^{\sigma\kappa}_{\varepsilon}$ $(*)$.

By IH(1) on premise $\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} \mathtt{w_i} \gg \tau_{\mathtt{i}}$, we get $(m, \theta(\mathtt{w_i})) \in [\![\sigma\tau_{\mathtt{i}}]\!]_{\mathtt{v}}$.

By (Fundamental) Theorem 19. on premise $\Delta; \Phi; \overline{x_i : \tau_i}, \Gamma \vdash_{\mathbb{C}} e : \tau \mid \kappa$ using

$\sigma \in \mathcal{D}[\![\Delta]\!]$,

$(m, \theta[x_i \mapsto \theta(\mathtt{w_i})]) \in \mathcal{G}[\![\mathtt{x_i} : \sigma\tau_{\mathtt{i}}, \sigma\Gamma]\!]$ (since $(m, \theta(\mathtt{w_i})) \in [\![\sigma\tau_{\mathtt{i}}]\!]_{\mathtt{v}}$ and $(m, \theta) \in \mathcal{G}[\![\sigma\Gamma]\!]$) and

$\models \sigma\Phi$,

we get $(m, \theta[x_i \mapsto \theta(\mathtt{w_i})]\mathtt{ee}) \in [\![\sigma\tau]\!]^{\sigma\kappa}_{\varepsilon}$ which is the same as $(*)$.

□

Corollary 21 (Type soundness for from-scratch execution)
Suppose that

$x : \tau \vdash_{\mathbb{C}} e : \tau' \mid \kappa$

$\vdash_{\mathbb{C}} v : \tau \mid -$

Then the following hold for some v', D and f:

1: $e[v/x] \Downarrow \langle v', D\rangle, f$

2: $f \leq \kappa$.

Proof. Immediate from the fundamental theorem (Theorem 45) statement (2), choosing any step index m greater than f. □

Corollary 22 (Type soundness for change propagation)
Suppose:

$x : \tau \vdash_{\mathbb{S}} e : \tau' \mid \kappa$

$\vdash_{\mathbb{S}} \mathtt{w} \gg \tau$

$e[\mathrm{L}(\mathtt{w})/\mathtt{x}] \Downarrow \mathtt{T}, \mathtt{f}$

Then, there exist T', c and $\mathtt{w}'$ such that

1: $\langle T, \ulcorner e \urcorner[\mathtt{w}/\mathtt{x}]\rangle \curvearrowright \mathtt{w}', \mathtt{T}', \mathtt{c}'$

2: $e[\mathrm{R}(\mathtt{w})/\mathtt{x}] \Downarrow \mathtt{T}', \mathtt{f}'$ and $\mathtt{V}(T') = \mathrm{R}(\mathtt{w}')$

3: $c' \leq \kappa$

Proof. Immediate from the fundamental theorem (Theorem 45) statement (1), choosing any step index m greater than f. □

$$\text{Types } \tau ::= \texttt{real} \mid \texttt{int} \mid \texttt{unit} \mid \tau_1 + \tau_2 \mid \tau_1 \times \tau_2 \mid \texttt{reflist}\ \tau \mid \tau_1 \to \tau_2 \mid \texttt{ref}\ \tau$$

Figure 23: Target Types

$$\begin{aligned}\text{Target expressions } e, f ::= {} & x \mid r \mid () \mid \bot \\ & \mid (e_1, e_2) \mid \texttt{fst}\ e \mid \texttt{snd}\ e \\ & \mid 0 \mid \texttt{succ}\ e \mid \texttt{case}_{\texttt{N}}\ e\ \texttt{of}\ 0 \to e_1 \mid \texttt{succ}(x) \to e_2 \\ & \mid \texttt{inl}\ e \mid \texttt{inr}\ e \mid \texttt{case}(e, x.y, e_1.e_2) \\ & \mid \texttt{nil} \mid \texttt{cons}(e_1, e_2) \mid (\texttt{case}_{\texttt{L}}\ e\ \texttt{of}\ [] \to e_1 \mid \texttt{cons}(h, tl) \to e_2) \\ & \mid \texttt{fix}\ f(x).\, e \mid e_1\ e_2 \mid \texttt{let}\, x = e_1\, \texttt{in}\, e_2 \mid \zeta\ e \\ & \mid l \mid \texttt{ref}\ e \mid !e \\ & \mid \texttt{read}(e_1,\ x.\ e_2)\end{aligned}$$

Figure 24: Target Language

$$\boxed{e,\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} v,\ \sigma',\ t_2,\ c}$$

$$\frac{\begin{array}{c} e_1,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} l,\ \sigma',\ t_2,\ c_1 \\ l_n = \mathtt{fresh}_L(\sigma') \qquad \sigma'(l) = (v',\ \vec{e}) \qquad e_2[v'/x],\ \sigma'[l_n \mapsto \square],\ t_2 + 1 \Downarrow^{\mathbb{C}(l_n)}_{L,\beta} v,\ \sigma'',\ t_3,\ c_2 \end{array}}{\mathtt{read}(e_1,\ x.\ e_2),\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} l_n,\ \sigma''[l \mapsto (v',\ (l_n,\ \lambda x.\, e_2,\ t_2,\ t_3) :: \vec{e}),\ l_n \mapsto (v,\ [])],\ t_3 + 1,\ c_1 + c_2 + 1}\mathbf{read}_{\mathbb{S}}$$

$$\frac{e_1,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} l,\ \sigma',\ t_2,\ c_1 \qquad \sigma'(l) = (v',\ \vec{e}) \qquad e_2[v'/x],\ \sigma',\ t_2 + 1 \Downarrow^{\mathbb{C}(l_n)}_{L,\beta} v,\ \sigma'',\ t_3,\ c}{\mathtt{read}(e_1,\ x.\ e_2),\ \sigma,\ t_1 \Downarrow^{\mathbb{C}(l_n)}_{L,\beta} v,\ \sigma''[l \mapsto (v',\ (l_n,\ \lambda x.\, e_2,\ t_2,\ t_3) :: \vec{e})],\ t_3 + 1,\ c_1 + c_2 + 1}\mathbf{read}_{\mathbb{C}}$$

$$\frac{\begin{array}{c} e_1,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} \mathtt{fix}\ f(x).\, e,\ \sigma',\ t_2,\ c_1 \\ e_2,\ t_2,\ \sigma' \Downarrow^{\mathbb{S}}_{L,\beta} v',\ \sigma'',\ t_3,\ c_2 \qquad e[e/f][v'/x],\ \sigma'',\ t_3 \Downarrow^{r}_{L,\beta} v,\ \sigma''',\ t_4,\ c_3 \end{array}}{e_1\ e_2,\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} v,\ \sigma''',\ t_4,\ c_1 + c_2 + c_3 + 1}\mathbf{app}$$

$$\frac{e_1,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} v_1,\ \sigma',\ t_2,\ c_1 \qquad e_2,\ \sigma',\ t_2 \Downarrow^{\mathbb{S}}_{L,\beta} v_3,\ \sigma'',\ t_3,\ c_1}{(e_1, e_2),\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} (v_1, v_2),\ \sigma'',\ t_3,\ c_1 + c_2}\mathbf{pair}$$

$$\frac{e,\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} (v_1,\ v_2),\ \sigma',\ t_2,\ c}{\mathtt{fst}\ e,\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} v_1,\ \sigma',\ t_2,\ c + 1}\mathbf{fst} \qquad \frac{e,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} v,\ \sigma',\ t_2,\ c \qquad l = \mathtt{fresh}_L(\sigma')}{\mathtt{ref}\ e,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} l,\ \sigma'[l \mapsto v],\ t_2,\ c + 1}\mathbf{ref}_{\mathbb{S}}$$

$$\frac{e,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} v,\ \sigma',\ t_2,\ c}{\mathtt{ref}\ e,\ \sigma,\ t_1 \Downarrow^{\mathbb{C}(l)}_{L,\beta} v,\ \sigma',\ t_2,\ c + 1}\mathbf{ref}_{\mathbb{C}} \qquad \frac{e,\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} l,\ \sigma',\ t_2,\ c \qquad \sigma'(\beta(l)) = (v,\ \_)}{!e,\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} v,\ \sigma',\ t_2,\ c + 1}\mathbf{deref}$$

$$\frac{e_1,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} v_1,\ \sigma',\ t_2,\ c_1 \qquad e_2[v_1/x],\ \sigma',\ t_2 \Downarrow^{r}_{L,\beta} v_2,\ \sigma'',\ t_3,\ c_2}{\mathtt{let}\, x\ =\ e_1\, \mathtt{in}\ e_2,\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} v_2,\ \sigma'',\ t_2,\ c_1 + c_2 + 1}\mathbf{let}$$

$$\frac{e,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} v,\ \sigma',\ t_2,\ c}{\mathtt{inl}\ e,\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} \mathtt{inl}\ v,\ \sigma',\ t_2,\ c}\mathbf{inl}$$

$$\frac{e,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} \mathtt{inl}\ v',\ \sigma',\ t_2,\ c_1 \qquad e_1[v'/x],\ \sigma',\ t_2 \Downarrow^{r}_{L,\beta} v,\ \sigma'',\ t_3,\ c_2}{\mathtt{case}(e, x.e_1, y.e_2),\ \sigma,\ t_1 \Downarrow^{r}_{L,\beta} v,\ \sigma'',\ t_2,\ c_1 + c_2 + 1}\mathbf{case}$$

Figure 25: Subset of the evaluation semantics

$$\begin{array}{lcl}
\|(A)^{\mu}\| & = & \mathtt{ref}\ \|A\|_A \\
\|\square(\tau)\| & = & \|\tau\| \\
\|\exists i{::}S.\ \tau\| & = & \|\tau\| \\
\|C \supset \tau\| & = & \|\tau\| \\
\|C\ \&\ \tau\| & = & \|\tau\| \\
\|\mathtt{int}\|_A & = & \mathtt{int} \\
\|\tau_1 + \tau_2\|_A & = & \|\tau_1\| + \|\tau_2\| \\
\|\tau_1 \times \tau_2\|_A & = & \|\tau_1\| \times \|\tau_2\| \\
\|\mathtt{list}\,[n]^{\alpha}\ \tau\|_A & = & \mathtt{reflist}\ \|\tau\| + \|\tau\| \\
\|\tau_1 \xrightarrow{\delta(\kappa)} \tau_2\|_A & = & \|\tau_1\| \to \|\tau_2\| \\
\|\forall i \overset{\delta(\kappa)}{::} S.\ \tau\|_A & = & \mathtt{unit} \to \|\tau\| \\
\|\mathtt{unit}\|_A & = & \mathtt{unit}
\end{array}$$

Figure 26: Translation of types

Definition 1 (Partial application)

$\boxed{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner}$

$$\frac{}{\Delta; \Phi; \Gamma, x : \tau \vdash_{\mathbb{S}} x : \tau \mid 0 \hookrightarrow x}\ \mathbf{var}_{\mathbb{S}} \qquad \frac{}{\Delta; \Phi; \Gamma, x : \tau \vdash_{\mathbb{C}} x : \tau \mid c_{var}() \hookrightarrow \mathtt{read}(x,\ x.\ x)}\ \mathbf{var}_{\mathbb{C}}$$

$$\frac{\Delta; \Phi \vdash \Gamma\ \mathtt{wf} \qquad \kappa = (\epsilon \doteq \mathbb{C}\ ?\ c_{real}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon r : (\mathtt{real})^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ r}\ \mathbf{real}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : \tau_1 \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta; \Phi; \Gamma \vdash_\epsilon e_2 : \tau_2 \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \qquad \kappa = \kappa_1 + \kappa_2 + (\epsilon \doteq \mathbb{C}\ ?\ c_{pair}()\ :\ 0)\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon (e_1, e_2) : (\tau_1 \times \tau_2)^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ (\ulcorner e_1 \urcorner, \ulcorner e_2 \urcorner)}\ \mathbf{pair}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : (\tau_1 \times \tau_2)^{\mathbb{S}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = \kappa' + c_{fst}(\epsilon, \mathbb{S})}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{fst}\ e : \tau_1 \mid \kappa \hookrightarrow \mathtt{fst}\ !\ulcorner e \urcorner}\ \mathbf{fst}_{\mathbb{S}}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : (\tau_1 \times \tau_2)^{\mathbb{S}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = \kappa' + c_{snd}(\epsilon, \mathbb{S})}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{snd}\ e : \tau_2 \mid \kappa \hookrightarrow \mathtt{snd}\ !\ulcorner e \urcorner}\ \mathbf{snd}_{\mathbb{S}}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : (\tau_1 \times \tau_2)^{\mathbb{C}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = \kappa' + c_{fst}(\epsilon, \mathbb{C}) \qquad \models \mathbb{C} \trianglelefteq \tau_1}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{fst}\ e : \tau_1 \mid \kappa \hookrightarrow \mathtt{read}(\ulcorner e \urcorner,\ x.\ \mathtt{fst}\ x)}\ \mathbf{fst}_{\mathbb{C}}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : (\tau_1 \times \tau_2)^{\mathbb{C}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = \kappa' + c_{snd}(\epsilon, \mathbb{C}) \qquad \models \mathbb{C} \trianglelefteq \tau_2}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{snd}\ e : \tau_2 \mid \kappa \hookrightarrow \mathtt{read}(\ulcorner e \urcorner,\ x.\ \mathtt{snd}\ x)}\ \mathbf{snd}_{\mathbb{C}}$$

$$\frac{\Delta; \Phi; \Gamma, f : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}}, x : \tau_1 \vdash_\delta e : \tau_2 \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = (\epsilon = \mathbb{C}\ ?\ c_{fix}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{fix}\ f(x).\, e : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ (\mathtt{fix}\ f(x).\ulcorner e \urcorner)}\ \mathbf{fix1}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}} \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta; \Phi; \Gamma \vdash_\epsilon e_2 : \tau_1 \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \qquad \epsilon \leq \delta \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + c_{app}(\epsilon, \mathbb{S})\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon e_1\ e_2 : \tau_2 \mid \kappa \hookrightarrow\ !\ulcorner e_1 \urcorner\ \ulcorner e_2 \urcorner}\ \mathbf{app}_{\mathbb{S}}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : (\tau_1 \xrightarrow{\mathbb{C}(\kappa')} \tau_2)^{\mathbb{C}} \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta; \Phi; \Gamma \vdash_\epsilon e_2 : \tau_1 \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \qquad \models \mathbb{C} \trianglelefteq \tau_2 \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + c_{app}(\epsilon, \mathbb{C})\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon e_1\ e_2 : \tau_2 \mid \kappa \hookrightarrow \mathtt{let}\, f\ =\ \ulcorner e_1 \urcorner\, \mathtt{in}\, \mathtt{let}\, x\ =\ \ulcorner e_2 \urcorner\, \mathtt{in}\, \mathtt{read}(f,\ f.\ f\ x)}\ \mathbf{app}_{\mathbb{C}}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau_1 \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = \kappa' + (\epsilon \doteq \mathbb{C}\ ?\ c_{inl}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{inl}\ e : (\tau_1 + \tau_2)^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ (\mathtt{inl}\ \ulcorner e \urcorner)}\ \mathbf{inl}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau_2 \mid \kappa \hookrightarrow \ulcorner e \urcorner \qquad \kappa = \kappa' + (\epsilon \doteq \mathbb{C}\ ?\ c_{inr}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{inr}\ e : (\tau_1 + \tau_2)^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ (\mathtt{inr}\ \ulcorner e \urcorner)}\ \mathbf{inr}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e : (\tau_1 + \tau_2)^{\mathbb{S}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \\ \Delta; \Phi; \Gamma, x : \tau_1 \vdash_\epsilon e_1 : \tau \mid \kappa' \hookrightarrow \ulcorner e_1 \urcorner \qquad \Delta; \Phi; \Gamma, y : \tau_2 \vdash_\epsilon e_2 : \tau \mid \kappa' \hookrightarrow \ulcorner e_2 \urcorner \qquad \kappa = \kappa_e + \kappa' + c_{case}(\epsilon, \mathbb{S})\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{case}(e, x.e_1, y.e_2) : \tau \mid \kappa \hookrightarrow \mathtt{case}(!\ulcorner e \urcorner, x.\ulcorner e_1 \urcorner, y.\ulcorner e_2 \urcorner)}\ \mathbf{case}_{\mathbb{S}}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e : (\tau_1 + \tau_2)^{\mathbb{C}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \qquad \Delta; \Phi; \Gamma, x : \tau_1 \vdash_{\mathbb{C}} e_1 : \tau \mid \kappa' \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta; \Phi; \Gamma, y : \tau_2 \vdash_{\mathbb{C}} e_2 : \tau \mid \kappa' \hookrightarrow \ulcorner e_2 \urcorner \qquad \models \mathbb{C} \trianglelefteq \tau \qquad \kappa = \kappa_e + \kappa' + c_{case}(\epsilon, \mathbb{C})\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{case}(e, x.e_1, y.e_2) : \tau \mid \kappa \hookrightarrow \mathtt{read}(\ulcorner e \urcorner,\ w.\ \mathtt{case}(w, x.e_1, y.e_2))}\ \mathbf{case}_{\mathbb{C}}$$

$$\frac{\kappa = (\epsilon = \mathbb{C}\ ?\ c_{zero}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{0} : (\mathtt{nat}[0])^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ 0}\ \mathbf{zero}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e : (\mathtt{nat}[n])^{\mathbb{S}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \\ \kappa = \kappa' + (\epsilon = \mathbb{C}\ ?\ c_{succ}()\ :\ 0)\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{succ}\ e : (\mathtt{nat}[n+1])^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ (\mathtt{succ}\ \ulcorner e \urcorner)}\ \mathbf{succ}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e : (\mathtt{nat}[n])^{\mathbb{S}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \qquad \Delta; \Phi \wedge n \doteq 0; \Gamma \vdash_\epsilon e_1 : (\mathtt{nat}[n])^{\mathbb{S}} \mid \kappa' \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta, i :: \iota; \Phi \wedge n \doteq i+1; \Gamma, x : \mathtt{nat}[i] \vdash_\epsilon e_1 : (\mathtt{nat}[n])^{\mathbb{S}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = \kappa_e + \kappa' + (\epsilon = \mathbb{C}\ ?\ c_{caseN}()\ :\ 0)\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{case_N}\ e\ \mathtt{of}\ 0\ \rightarrow\ e_1 \mid \mathtt{succ}(x)\ \rightarrow\ e_2 : \tau \mid \kappa \hookrightarrow \mathtt{case_N}\ !\ulcorner e \urcorner\ \mathtt{of}\ 0\ \rightarrow\ \ulcorner e_1 \urcorner \mid \mathtt{succ}(x)\ \rightarrow\ \ulcorner e_2 \urcorner}\ \mathbf{caseN}$$

Figure 27: Translation rules

$\boxed{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa}$ expression e has type τ with dynamic stability κ. The context Υ carrying types of primitive functions is omitted from all rules.

$$\frac{\Upsilon(\zeta) = \zeta : (B_1 \dots B_n) \xrightarrow{\kappa'} B \qquad \Delta;\Phi;\Gamma \vdash_\epsilon e_i : (B_i)^{\mu_i} \mid \kappa_{e_i} \hookrightarrow \ulcorner e_i \urcorner \qquad \mu_1 \sqcup \dots \sqcup \mu_n = \mathbb{S} \qquad \kappa = (\sum_{i=1}^{n} \kappa_{e_i}) + (\epsilon = \mathbb{C}\ ?\ \kappa' :\ 0) + c_{prim}(\epsilon, n, \mu_1, \dots, \mu_n)}{\Delta;\Phi;\Gamma \vdash_\epsilon \zeta\ e_1 \dots e_n : (B)^{\mathbb{S}} \mid \kappa \hookrightarrow \texttt{ref}\ \zeta\ !\ulcorner e_1 \urcorner \dots !\ulcorner e_n \urcorner}\ \textbf{primApp}_{\mathbb{S}}$$

$$\frac{\Upsilon(\zeta) = \zeta : (B_1 \dots B_n) \xrightarrow{\kappa'} B \qquad \Delta;\Phi;\Gamma \vdash_\epsilon e_i : (B_i)^{\mu_i} \mid \kappa_{e_i} \hookrightarrow \ulcorner e_i \urcorner \qquad \mu_1 \sqcup \dots \sqcup \mu_n = \mathbb{C} \qquad \vec{x} = \{\ x_i \mid \mu_i = \mathbb{C}\ \} \qquad x'_i = (\mu_i = \mathbb{C}\ ?\ x_i :\ !x_i) \qquad \kappa = (\sum_{i=1}^{n} \kappa_{e_i}) + +\kappa' + c_{prim}(\epsilon, n, \mu_1, \dots, \mu_n)}{\Delta;\Phi;\Gamma \vdash_\epsilon \zeta\ e_1 \dots e_n : (B)^{\mathbb{C}} \mid \kappa \hookrightarrow \texttt{let}\ x_i\ =\ \ulcorner e_i \urcorner\ \texttt{in}\ \texttt{read}(\vec{x},\ \vec{x}.\ \zeta\ x'_1 \dots x'_n)}\ \textbf{primApp}_{\mathbb{C}}$$

$$\frac{\Delta, t :: S;\Phi;\Gamma \vdash_\mu e : \tau \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa' = (\epsilon = \mathbb{C}\ ?\ c_{ifun}() :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \Lambda.\, e : (\forall t \overset{\mu(\kappa')}{::} S.\ \tau)^{\mathbb{S}} \mid \kappa \hookrightarrow \texttt{ref}\ \lambda().\ulcorner e \urcorner}\ \forall\textbf{I}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e : (\forall t \overset{\delta(\kappa')}{::} S.\ \tau)^{\mathbb{S}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \qquad \Delta \vdash I :: S \qquad \epsilon \le \delta \qquad \kappa = \kappa_e + \kappa'\{I/t\} + c_{iApp}(\epsilon, \mathbb{S})}{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e[] : \tau\{I/t\} \mid \kappa \hookrightarrow\ !\ulcorner e \urcorner\ ()}\ \forall\textbf{E}_{\mathbb{S}}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : (\forall t \overset{\mathbb{C}(\kappa')}{::} S.\ \tau)^{\mathbb{C}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \qquad \Delta \vdash I :: S \qquad \models \mathbb{C} \trianglelefteq \tau\{I/t\} \qquad \kappa = \kappa_e + \kappa'\{I/t\} + c_{iApp}(\epsilon, \mathbb{C})}{\Delta;\Phi;\Gamma \vdash_\epsilon e[] : \tau\{I/t\} \mid \kappa \hookrightarrow \texttt{read}(\ulcorner e \urcorner,\ f.\ f\ ())}\ \forall\textbf{E}_{\mathbb{C}}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau\{I/t\} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \Delta \vdash I :: S \qquad \kappa = \kappa' + c_{pack}()}{\Delta;\Phi;\Gamma \vdash_\epsilon \texttt{pack}\ e : (\exists t{::}S.\ \tau)^{\mathbb{S}} \mid \kappa \hookrightarrow \texttt{ref}\ \ulcorner e \urcorner}\ \exists\textbf{I}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : (\exists t{::}S.\ \tau)^{\mathbb{S}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \qquad \Delta, t :: S;\Phi;\Gamma, x : \tau \vdash_\epsilon e' : \tau' \mid \kappa' \hookrightarrow \ulcorner e' \urcorner \qquad \kappa = \kappa_e + \kappa' + c_{unpack}(\epsilon, \mathbb{S})}{\Delta;\Phi;\Gamma \vdash_\epsilon \texttt{unpack}\ e\ \texttt{as}\ x\ \texttt{in}\ e' : \tau' \mid \kappa \hookrightarrow \texttt{let}\ x\ =\ !\ulcorner e \urcorner\ \texttt{in}\ \ulcorner e' \urcorner}\ \exists\textbf{E}_{\mathbb{S}}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : (\exists t{::}S.\ \tau)^{\mathbb{C}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \qquad \Delta, t :: S;\Phi;\Gamma, x : \tau \vdash_{\mathbb{C}} e' : \tau' \mid \kappa' \hookrightarrow \ulcorner e' \urcorner \qquad \kappa = \kappa_e + \kappa' + c_{unpack}(\epsilon, \mathbb{C}) \qquad \models \tau \trianglelefteq \mathbb{C}}{\Delta;\Phi;\Gamma \vdash_\epsilon \texttt{unpack}\ e\ \texttt{as}\ x\ \texttt{in}\ e' : \tau' \mid \kappa \hookrightarrow \texttt{let}\ y\ =\ \ulcorner e \urcorner\ \texttt{in}\ \texttt{read}(y,\ y.\ \texttt{let}\ x\ =\ y\ \texttt{in}\ \ulcorner e' \urcorner)}\ \exists\textbf{E}_{\mathbb{C}}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \Delta;\Phi \wedge C \models \kappa' \le \kappa \qquad \forall x \in \Gamma\ \ \Delta;\Phi \wedge \neg C \models \Gamma(x) \sqsubseteq \square(\Gamma(x))}{\Delta;\Phi;\Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner}\ \textbf{split}$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau' \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \Delta;\Phi \models \tau' \sqsubseteq \tau \qquad \Delta;\Phi \models \kappa' \le \kappa}{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner}\ \sqsubseteq$$

$$\frac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \forall x \in \Gamma,\ \ \Delta;\Phi \models \Gamma(x) \sqsubseteq \square(\Gamma(x)) \qquad \kappa = (\epsilon = \mathbb{C}\ ?\ \kappa' :\ 0)}{\Delta;\Phi;\Gamma, \Gamma' \vdash_\epsilon e : \square(\tau) \mid \kappa \hookrightarrow \ulcorner e \urcorner}\ \textbf{nochange}$$

$$\frac{\Delta;\Phi;\Gamma, f : \square((\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}}, x : \tau_1) \vdash_\delta e : \tau_2 \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \forall x \in \Gamma,\ \ \Delta;\Phi \models\!\models \Gamma(x) \sqsubseteq \square(\Gamma(x)) \qquad \kappa = (\epsilon = \mathbb{C}\ ?\ c_{fix}() :\ 0)}{\Delta;\Phi;\Gamma \vdash_\epsilon \texttt{fix}\ f(x).\, e : \square((\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}}) \mid \kappa \hookrightarrow \texttt{ref}\ (\texttt{fix}\ f(x).\ulcorner e \urcorner)}\ \textbf{fix2}$$

Figure 28: Translation rules, part 2

$\boxed{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner}$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : \tau_1 \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \quad \Delta; \Phi; \Gamma, x : \tau_1 \vdash_\epsilon e_2 : \tau_2 \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \quad \kappa = \kappa_1 + \kappa_2 + (\epsilon = \mathbb{C}\ ?\ c_{let}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{let}\, x\ =\ e_1\, \mathtt{in}\, e_2 : \tau_2 \mid \kappa \hookrightarrow \mathtt{let}\, x\ =\ \ulcorner e_1 \urcorner\, \mathtt{in}\, \ulcorner e_2 \urcorner}\ \textbf{let}$$

$$\frac{\kappa = (\epsilon = \mathbb{C}\ ?\ c_{unit}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon () : (\mathtt{unit})^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ ()}\ \textbf{unit}$$

$$\frac{\Delta; \Phi \wedge C; \Gamma \vdash_\epsilon e : \tau \mid \kappa' \hookrightarrow \ulcorner e \urcorner \quad \kappa = \kappa' + (\epsilon = \mathbb{C}\ ?\ c_{impl}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon e : (C \supset \tau)^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ \ulcorner e \urcorner}\ \textbf{c-impI}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : (C \supset \tau)^{\mathbb{S}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \quad \Delta; \Phi \models C \quad \kappa = \kappa' + c_{dot}(\epsilon, \mathbb{S})}{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow\ !\ulcorner e \urcorner}\ \textbf{c-impE}_{\mathbb{S}}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : (C \supset \tau)^{\mathbb{C}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \quad \Delta; \Phi \models C \quad \kappa = \kappa' + c_{dot}(\epsilon, \mathbb{C})}{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \mathtt{read}(\ulcorner e \urcorner,\ x.\ x)}\ \textbf{c-impE}_{\mathbb{C}}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner \quad \Delta; \Phi \models C \quad \kappa = \kappa' + (\epsilon = \mathbb{C}\ ?\ c_{and}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon e : (C\ \&\ \tau)^{\mathbb{S}} \mid \kappa' \hookrightarrow \mathtt{ref}\ \ulcorner e \urcorner}\ \textbf{c-andI}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : (C\ \&\ \tau_1)^{\mathbb{S}} \mid \kappa \hookrightarrow \ulcorner e_1 \urcorner \quad \Delta; \Phi \wedge C; \Gamma, x : \tau_1 \vdash_\epsilon e_2 : \tau_2 \mid \kappa \hookrightarrow \ulcorner e_2 \urcorner \quad \kappa = \kappa' + c_{letAs}(\epsilon, \mathbb{S})}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{let}\, x\ =\ e_1\, \mathtt{in}\, e_2 : \tau_2 \mid \kappa \hookrightarrow \mathtt{clet}\ !\ulcorner e_1 \urcorner\, \mathtt{as}\, x\, \mathtt{in}\, \ulcorner e_2 \urcorner}\ \textbf{c-andE}_{\mathbb{S}}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : (C\ \&\ \tau_1)^{\mathbb{C}} \mid \kappa \hookrightarrow \ulcorner e_1 \urcorner \quad \Delta; \Phi \wedge C; \Gamma, x : \tau_1 \vdash_{\mathbb{C}} e_2 : \tau_2 \mid \kappa \hookrightarrow \ulcorner e_2 \urcorner \quad \kappa = \kappa' + c_{letAs}(\epsilon, \mathbb{C})}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{let}\, x\ =\ e_1\, \mathtt{in}\, e_2 : \tau_2 \mid \kappa \hookrightarrow \mathtt{read}(\ulcorner e_1 \urcorner,\ x.\ \ulcorner e_2 \urcorner)}\ \textbf{c-andE}_{\mathbb{C}}$$

$$\frac{\Delta; \Phi \models \bot}{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \bot}\ \textbf{contra} \qquad \frac{\kappa = (\epsilon = \mathbb{C}\ ?\ c_{nil}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{nil} : (\mathtt{list}\, [0]^0\ \tau)^{\mathbb{S}} \mid \kappa \hookrightarrow \mathtt{ref}\ \mathtt{nil}}\ \textbf{nil}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : \Box(\tau) \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \quad \Delta; \Phi; \Gamma \vdash_\epsilon e_2 : (\mathtt{list}\, [n]^\alpha\ \tau)^\mu \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \quad \kappa = \kappa_1 + \kappa_2 + (\epsilon = \mathbb{C}\ ?\ c_{cons}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{cons}(e_1, e_2) : (\mathtt{list}\, [n{+}1]^\alpha\ \tau)^\mu \mid \kappa \hookrightarrow \mathtt{ref}\ (\mathtt{cons}(\mathtt{inl}\ \ulcorner e_1 \urcorner, \ulcorner e_2 \urcorner))}\ \textbf{cons1}$$

$$\frac{\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : \tau \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \quad \Delta; \Phi; \Gamma \vdash_\epsilon e_2 : (\mathtt{list}\, [n]^{\alpha-1}\ \tau)^\mu \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \quad \Delta; \Phi \models \alpha > 0 \quad \kappa = \kappa_1 + \kappa_2 + (\epsilon = \mathbb{C}\ ?\ c_{cons}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{cons}(e_1, e_2) : (\mathtt{list}\, [n{+}1]^\alpha\ \tau)^\mu \mid \kappa \hookrightarrow \mathtt{ref}\ (\mathtt{cons}(\mathtt{inr}\ \ulcorner e_1 \urcorner, \ulcorner e_2 \urcorner))}\ \textbf{cons2}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e : (\mathtt{list}\, [n]^\alpha\ \tau)^{\mathbb{S}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \quad \Delta; \Phi \wedge n \doteq 0; \Gamma \vdash_\epsilon e_1 : \tau' \mid \kappa' \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta, i :: \iota; \Phi \wedge n \doteq i + 1; \Gamma, h : \Box(\tau), tl : (\mathtt{list}\, [i]^\alpha\ \tau)^{\mathbb{S}} \vdash_\epsilon e_2 : \tau' \mid \kappa' \hookrightarrow \ulcorner e_{2_l} \urcorner \\ \Delta, i :: \iota, \beta :: \iota; \Phi \wedge n \doteq i + 1 \wedge \alpha \doteq \beta + 1; \Gamma, h : \tau, tl : (\mathtt{list}\, [i]^\beta\ \tau)^{\mathbb{S}} \vdash_\epsilon e_2 : \tau' \mid \kappa' \hookrightarrow \ulcorner e_{2_r} \urcorner \\ \kappa = \kappa_e + \kappa' + c_{caseL}(\epsilon, \mathbb{S})\end{array}}{\begin{array}{l}\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{case_L}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2 : \tau' \mid \kappa \hookrightarrow \\ \qquad\qquad \mathtt{case_L}\ !\ulcorner e \urcorner\ \mathtt{of} \\ \qquad\qquad \mid\ \mathtt{nil}\ \rightarrow\ \ulcorner e_1 \urcorner \\ \qquad\qquad \mid\ \mathtt{cons}(h, tl)\ \rightarrow\ \mathtt{case}(h, h.\ulcorner e_{2_l} \urcorner, h.\ulcorner e_{2_r} \urcorner)\end{array}}\ \textbf{caseL}_{\mathbb{S}}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e : (\mathtt{list}\, [n]^\alpha\ \tau)^{\mathbb{C}} \mid \kappa_e \hookrightarrow \ulcorner e \urcorner \quad \Delta; \Phi \wedge n \doteq 0; \Gamma \vdash_{\mathbb{C}} e_1 : \tau' \mid \kappa' \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta, i :: \iota; \Phi \wedge n \doteq i + 1; \Gamma, h : \Box(\tau), tl : (\mathtt{list}\, [i]^\alpha\ \tau)^{\mathbb{S}} \vdash_{\mathbb{C}} e_2 : \tau' \mid \kappa' \hookrightarrow \ulcorner e_{2_l} \urcorner \\ \Delta, i :: \iota, \beta :: \iota; \Phi \wedge n \doteq i + 1 \wedge \alpha \doteq \beta + 1; \Gamma, h : \tau, tl : (\mathtt{list}\, [i]^\beta\ \tau)^{\mathbb{S}} \vdash_{\mathbb{C}} e_2 : \tau' \mid \kappa' \hookrightarrow \ulcorner e_{2_r} \urcorner \\ \models \mathbb{C} \trianglelefteq \tau \qquad \kappa = \kappa_e + \kappa' + c_{caseL}(\epsilon, \mathbb{C})\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon \mathtt{case_L}\ e\ \mathtt{of\ nil}\ \rightarrow\ e_1 \mid \mathtt{cons}(h, tl)\ \rightarrow\ e_2 : \tau' \mid \kappa \hookrightarrow \mathtt{read}(\ulcorner e \urcorner,\ x.\ f)}\ \textbf{caseL}_{\mathbb{C}}$$

where $f = \mathtt{case_L}\ x\ \mathtt{of\ nil}\ \rightarrow\ \ulcorner e_1 \urcorner \mid \mathtt{cons}(h, tl)\ \rightarrow\ \mathtt{case}(h, h.\ulcorner e_{2_l} \urcorner, h.\ulcorner e_{2_r} \urcorner)$

Figure 29: Translation rules, part 3

Let $\beta \subseteq S_1 \rightharpoonup S_2$ be a partial bijection. We define the following.

$$\beta(x) = \begin{cases} x & \text{if } x \notin \texttt{dom}(\beta) \\ y & \text{if } (x,\ y) \in \beta \end{cases}$$

Definition 2 (Heap well-formedness)

$$\texttt{wf}\ \varnothing \qquad\qquad \frac{\texttt{wf}\ \sigma \qquad l \notin \texttt{dom}(\sigma) \qquad \forall\ l' \in \texttt{FL}(v),\ l' \in \texttt{dom}(\sigma)}{\texttt{wf}\ \sigma[l \mapsto (v,\ \_)]}$$

Definition 3 (Heap reachability)

$$\frac{l_s \neq l \qquad \mathcal{R}_\sigma(l_s,\ l_d)}{\mathcal{R}_{\sigma[l \mapsto (v,\ \_)]}(l_s,\ l_d)} \qquad\qquad \frac{(l_d = l_s) \vee (\exists\ l \in \texttt{FL}(v),\ \mathcal{R}_\sigma(l,\ l_d))}{\mathcal{R}_{\sigma[l_s \mapsto (v,\ \_)]}(l_s,\ l_d)}$$

$$\mathcal{R}_\sigma(S) = \{\ l \mid \exists\ l' \in S.\ \mathcal{R}_\sigma(l',\ l)\ \}$$

Definition 4 (Heap extension)

$$\begin{array}{rcl} \sigma_2 \sqsupseteq \sigma_1 & = & (\forall l \in \texttt{dom}(\sigma_1), l \in \texttt{dom}(\sigma_2)) \wedge \\ & & (\forall l \in \texttt{dom}(\sigma_1),\ \sigma_1(l) = (v,\ \vec{e}) \Rightarrow \exists \vec{e'},\ \sigma_2(l) = (v,\ \vec{e'} +\!\!+ \vec{e})) \end{array}$$

Definition 5 (Heap edges)

$$\begin{array}{lcl} \texttt{edges}(\varnothing) & = & \emptyset \\ \texttt{edges}(\sigma[l \mapsto (v,\ (l_d,\ f,\ t_1,\ t_2) :: \vec{e})]) & = & (l_s,\ l_d,\ f,\ t_1,\ t_2) \cup \texttt{edges}(\sigma[l \mapsto (v,\ \vec{e})]) \\ \texttt{edges}(\sigma[l \mapsto (v,\ [])]) & = & \texttt{edges}(\sigma) \end{array}$$

Definition 6 (Path)

$$\frac{(l_s,\ l_d,\ f,\ t_1,\ t_2) \in S \qquad l_s \in R \vee l_d \in R}{\texttt{path}(R,\ S,\ l)} \qquad \frac{(l_s,\ l,\ f,\ t_1,\ t_2) \in S \qquad \texttt{path}(R,\ S,\ l_s)}{\texttt{path}(R,\ S,\ l)}$$

Definition 7 (Graph definitions)

$$\begin{array}{rcl} \texttt{trg}(\emptyset) & = & \emptyset \\ \texttt{trg}(\{(l_s,\ l_d,\ f,\ \_,\ \_)\} \cup S) & = & \{l_d\} \cup \texttt{trg}(S) \end{array}$$

$$\begin{array}{rcl} \texttt{src}([]) & = & \emptyset \\ \texttt{src}((l_s,\ l_d,\ f\ \_,\ \_) \cup S) & = & \{l_s\} \cup \texttt{src}(S) \end{array}$$

$$\begin{array}{rcl} \texttt{locs}([]) & = & \emptyset \\ \texttt{locs}((l_s,\ l_d,\ f\ \_,\ \_) \cup S) & = & \texttt{FL}(f) \cup \texttt{locs}(S) \end{array}$$

$$\begin{array}{rcl} \texttt{FL}([]) & = & \emptyset \\ \texttt{FL}((l_s,\ l_d,\ f\ \_,\ \_) \cup S) & = & \texttt{FL}(f) \cup \texttt{FL}(S) \end{array}$$

$$\texttt{locs}(S) \ = \ \texttt{trg}(S) \cup \texttt{src}(S)$$

Definition 8 (Dependency graph)

$$\mathcal{D}(S,\ R) = [(l_{1_s},\ l_{1_s},\ f_1,\ t_{1_i},\ t_{1_e}), \ldots,\ (l_{n_s},\ l_{n_d},\ f_n,\ t_{n_i},\ t_{n_e})]$$

such that $t_{1_i} \leq \cdots \leq t_{n_e}$, all elements are unique, and $(l_s,\ l_d,\ f,\ t_1,\ t_2) \in D(S,\ R)$ if and only if $(l_s,\ l_d,\ f,\ t_1,\ t_2) \in S$ and $\texttt{path}(R,\ S,\ l_s)$ and there is no $(l'_s,\ l'_d,\ f',\ t'_1,\ t'_2) \in S$ such that $\texttt{path}(R,\ S,\ l'_s)$ and $t'_1 < t_1,\ t_2 < t'_2$.

Definition 9 (Heap target reachability (shallow))

$$\frac{l_s \neq l \qquad \hat{\mathcal{R}}_{(\sigma,\ R)}(l_s,\ l_d)}{\hat{\mathcal{R}}_{(\sigma[l \mapsto (v,\ \_)],\ R)}(l_s,\ l_d)}$$

$$\frac{l_d \notin \mathtt{trg}(\mathtt{edges}(\sigma)) \ \vee \ \neg \mathtt{path}(R,\ \mathtt{edges}(\sigma),\ l_f) \qquad (\exists\ l \in \mathtt{FL}(v),\ \hat{\mathcal{R}}_{(\sigma,\ R)}(l,\ l_d))}{\hat{\mathcal{R}}_{(\sigma \mapsto l(v,\ \_),\ R)}(l_s,\ l_d)}$$

$$\frac{l_d \in \mathtt{trg}(\mathtt{edges}(\sigma)) \qquad \mathtt{path}(R,\ \mathtt{edges}(\sigma),\ l_f)}{\hat{\mathcal{R}}_{(\sigma[l_d \mapsto (v,\ \_)],\ R)}(l_d,\ l_d)}$$

$$\hat{\mathcal{R}}_{(\sigma,\ S)}(=)\{\ l \mid \exists\ l' \in S.\ \hat{\mathcal{R}}_{(\sigma,\ l')}(l,\ \ )\}$$

Definition 10 (Change Propagation)

$$\frac{}{[],\ \sigma_f,\ \sigma_c,\ \beta \rightsquigarrow \sigma_c,\ \beta,\ 0}\ \textbf{stop}$$

$$\frac{\sigma(l_s) = (v,\ \_) \qquad l'_n = \mathtt{fresh}_{L_2}(\sigma_f \uplus \sigma_c) \qquad e[v/x],\ \sigma_f \uplus \sigma_c[l'_n \mapsto \square],\ t_1 \Downarrow^{\mathbb{C}(l'_n)}_{L_2,\ \beta} v,\ \sigma_f \uplus \sigma'_f,\ t_2,\ c \qquad D,\ \sigma_f,\ \sigma'_f[l'_n \mapsto (v,\ [])],\ \beta \otimes l_n \mapsto l'_n \rightsquigarrow \sigma''_f,\ \beta',\ c'}{(l_s,\ l_d,\ f) :: D,\ \sigma_f,\ \sigma_c,\ \beta \rightsquigarrow \sigma''_f,\ \beta',\ c + c' + 1}\ \textbf{eval}$$

Lemma 23 (Reachable set containment)
Assume that $\mathtt{wf}\ \sigma$. The following hold.

1. If $\mathcal{R}_\sigma(l,\ l')$ then $l' \in \mathtt{dom}(\sigma)$
2. $\mathcal{R}_\sigma(S) \subseteq \mathtt{dom}(\sigma)$
3. If $S \subseteq \mathtt{dom}(\sigma)$ then $S \subseteq \mathcal{R}_\sigma(S)$

Proof of (1). By induction on $\mathtt{wf}\ \sigma$ and case analysis on the reachability derivation. □

Proof of (2). It follows directly from the definition and (1). □

Lemma 24 (Reachability under store extension)
Assume that $\mathtt{wf}\ \sigma$ and $\mathtt{wf}\ \sigma'$. The following hold.

1. If $\sigma' \sqsupseteq \sigma$ and $\mathcal{R}_\sigma(l,\ l')$ then $\mathcal{R}_{\sigma'}(l,\ l')$
2. If $S \subseteq \sigma$ and $\sigma' \sqsupseteq \sigma$ then $\mathcal{R}_{\sigma'}(S) = \mathcal{R}_{\sigma'}(S)$

Proof of (1). By induction on $\mathtt{wf}\ \sigma$ and case analysis on the reachability derivation. □

Proof of (2). It follows directly from the definition and (1). □

Lemma 25 (Path facts)
The following hold.

1. If $\mathtt{path}(R,\ S,\ l)$ then $\mathtt{path}(R,\ S \cup S',\ l)$.
2. If $\mathtt{path}(R,\ S,\ l)$ then $\mathtt{path}(R \cup R',\ S,\ l)$.
3. If $\mathtt{path}(R \cup R',\ S,\ l)$ and for all $l \in R'$, $\mathtt{path}(R,\ S,\ l)$ then $\mathtt{path}(R,\ S,\ l)$.
4. If $\mathtt{path}(R,\ S_1 \cup S_2,\ l)$, $l \in \mathtt{trg}(S)_1$ and for all $\mathtt{locs}(S_1) \cap \mathtt{trg}(S_2) = \emptyset$, then $\mathtt{path}(R,\ S,\ l)$.
5. If for all $l' \in R'$, $\mathtt{path}(R,\ S_1,\ l')$, then if $\mathtt{path}(R \cup R',\ S_2,\ l)$ then $\mathtt{path}(R,\ S_1 \cup S_2,\ l)$

$\text{World} = ((L_1 \uplus L_2) \rightharpoonup \text{Target Value} \times [\text{Edge}]) \times (L_1 \rightharpoonup L_2) \times \text{Step index}$
$\mathcal{V}(\!|\tau|\!), \mathcal{V}_A(\!|\tau|\!) \subseteq \text{Source Value} \times \text{Target Value} \times \text{World}$
$\mathcal{E}(\!|\tau|\!)^\kappa \subseteq \text{Source Expression} \times \text{Target Expression} \times \text{World}$

$(\sigma, \beta,\ j) > (\sigma',\ \beta'\ j') = \mathtt{wf}\ \sigma\ \wedge\ \sigma \sqsupseteq \sigma'\ \wedge\ \beta = \beta'\ \wedge\ j < j'$

$$
\begin{array}{ll}
\mathcal{V}(\!|(A)^\mu|\!) & = \{\ (v_s,\ \beta(l),\ (\sigma,\ \beta,\ k)) \mid (v_i,\ \sigma(l),\ (\sigma,\ \beta,\ k)) \in \mathcal{V}_A(\!|A|\!)\ \} \\
\mathcal{V}(\!|\Box(\tau)|\!) & = \mathcal{V}(\!|\tau|\!) \\
 & \\
\mathcal{V}_A(\!|\mathtt{unit}|\!) & = \{\ ((),\ (),\ W) \mid \top\ \} \\
\mathcal{V}_A(\!|\mathtt{real}|\!) & = \{\ (r,\ r,\ W) \mid \top\ \} \\
\mathcal{V}_A(\!|\tau_1 \times \tau_2|\!) & = \{\ ((v_{s1}, v_{s2}),\ (v_{t1}, v_{t2}),\ W) \mid (v_{s1},\ v_{t1},\ W) \in \mathcal{V}(\!|\tau_1|\!)\ \wedge\ (v_{s2},\ v_{t2},\ W) \in \mathcal{V}(\!|\tau_2|\!)\ \} \\
\mathcal{V}_A(\!|\mathtt{list}\,[n]^\alpha\ \tau|\!) & = \{\ (\mathtt{nil},\ \mathtt{nil},\ (\sigma,\ \beta,\ k)) \mid \models n \doteq 0\ \} \\
\mathcal{V}_A(\!|\mathtt{list}\,[n]^\alpha\ \tau|\!) & = \{\ (\mathtt{cons}(v_s,\ vs_s),\ \mathtt{cons}(\mathtt{inl}\ v_t,\ vs_t),\ W) \mid \\
 & \quad (v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau|\!)\ \wedge (vs_s,\ vs_t,\ W) \in \mathcal{V}(\!|\mathtt{list}\,[n-1]^\alpha\ \tau|\!) \wedge \models 0 < n\ \} \cup \\
 & \quad \{\ (\mathtt{cons}(v_s,\ vs_s),\ \mathtt{cons}(\mathtt{inr}\ v_t,\ vs_t),\ W) \mid \\
 & \quad (v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau|\!)\ \wedge (vs_s,\ vs_t,\ W) \in \mathcal{V}(\!|\mathtt{list}\,[n-1]^\alpha\ \tau|\!) \wedge \models 0 < n\ \} \\
\mathcal{V}_A(\!|\exists t{::}S.\ \tau|\!) & = \{\ (\mathtt{pack}\ v_s,\ v_t,\ W) \mid \exists I\ S,\ \vdash I :: S\ \wedge\ (v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau[I/t]|\!)\ \} \\
\mathcal{V}_A(\!|C \supset \tau|\!) & = \{\ (v_s,\ v_t,\ W) \mid \neg \models C\ \vee\ (v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau|\!)\ \} \\
\mathcal{V}_A(\!|C\ \&\ \tau|\!) & = \{\ (v_s,\ v_t,\ W) \mid \models C\ \wedge\ (v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau|\!)\ \} \\
\mathcal{V}_A(\!|\tau_1 + \tau_2|\!) & = \{\ (\mathtt{inl}\ v_s,\ \mathtt{inl}\ v_t,\ W) \mid (v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau_1|\!)\ \} \cup \\
 & \quad \{\ (\mathtt{inr}\ v_s,\ \mathtt{inr}\ v_t,\ W) \mid (v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau_2|\!)\ \} \\
\mathcal{V}_A(\!|\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \tau_2|\!) & = \{\ (\mathtt{fix}\ f(x).\, e_s,\ \mathtt{fix}\ f(x).\, e_t,\ W) \mid \top\ \} \\
\mathcal{V}_A(\!|\tau_1 \xrightarrow{\mathbb{C}(\kappa)} \tau_2|\!) & = \{\ (\mathtt{fix}\ f(x).\, e_s,\ \mathtt{fix}\ f(x).\, e_t,\ W) \mid \\
 & \quad \forall\ (\sigma_i,\ \beta,\ m) > W\ v_s\ v_t.\ (v_s,\ v_t,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{V}(\!|\tau_1|\!)\ \wedge\ \sigma_i(l) = \mathtt{fix}\ f(x).\, e_t \Rightarrow \\
 & \quad\ ([x \mapsto v_s, f \mapsto \mathtt{fix}\ f(x).\, e_s]e_s,\ [x \mapsto v_t, f \mapsto l]e_t,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{E}(\!|\tau_2|\!)^\kappa\ \} \\
\mathcal{V}_A(\!|\forall t \overset{\mathbb{S}(\kappa)}{::} S.\ \tau|\!) & = \{\ (\Lambda.\, e_s,\ \Lambda.\, e_t,\ W) \mid \top\ \} \\
\mathcal{V}_A(\!|\forall t \overset{\mathbb{C}(\kappa)}{::} S.\ \tau|\!) & = \{\ (\Lambda.\, e_s,\ \Lambda.\, e_t,\ W) \mid \forall I\ S,\ \vdash I :: S \Rightarrow (e_s,\ e_t,\ W) \in \mathcal{E}(\!|\tau[I/t]|\!)^{\kappa[I/t]}\ \} \\
 & \\
\mathcal{E}(\!|\tau|\!)^\kappa & = \{\ (e_s,\ e_t,\ (\sigma,\ \beta,\ k)) \mid \\
 & \quad (\forall \sigma'\ \beta'\ t_1\ L\ r,\ (\sigma',\ \beta',\ \kappa) > (\sigma,\ \beta,\ k) \wedge (\forall l_n,\ r = \mathbb{C}(l_n) \Rightarrow \sigma'_i(l_n) = \Box) \Rightarrow \\
 & \quad \exists v_s\ v_t\ \sigma'\ t_2\ c. \\
 & \quad (1).\quad e_s \Downarrow v_s\ \wedge \\
 & \quad (2).\quad e_t,\ \sigma',\ t_1 \Downarrow^r_{L,\beta} v_t,\ \sigma'',\ t_2,\ c\ \wedge \\
 & \quad (3).\quad c \leq \kappa\ \wedge \\
 & \quad (4).\quad r = \mathbb{S} \Rightarrow (v_s,\ v_t,\ (\sigma'',\ \beta,\ k - c)) \in \mathcal{V}(\!|\tau|\!) \\
 & \quad (5).\quad \forall l_n,\ r = \mathbb{C}(l_n) \Rightarrow (v_s,\ l_n,\ (\sigma''[l_n \mapsto v_t],\ \beta,\ k - c)) \in \mathcal{V}(\!|\tau|\!))\ \}
\end{array}
$$

Figure 30: Unary step-indexed interpretation of types (Concrete semantics)

$\text{World} = (\text{Loc}_1 \rightharpoonup \text{Target Value}) \times (\text{Loc}_2 \rightharpoonup \text{Target Value}) \times (\text{Loc}_1 \rightharpoonup \text{Loc}_1) \times \text{Step index} \times \text{Step index}$
$\mathcal{V}[\![\tau]\!], \mathcal{V}_A[\![\tau]\!] \subseteq \text{Source Value} \times \text{Source Value} \times \text{Target Value} \times \text{World} \times \text{Step index}$
$\mathcal{E}[\![\tau]\!]^\kappa \subseteq \text{Source Expression} \times \text{Source Expression} \times \text{Target Expression} \times \text{World} \times \text{Step index}$

$\mathtt{inv}(\sigma,\ \beta,\ v) = \forall l\ R,\ l \in \hat{\mathcal{R}}_{(\sigma,\ \beta)}(\mathtt{FL}(v)) \Rightarrow l \in \mathtt{trg}(\mathcal{D}(\mathtt{edges}(\sigma),\ R))$

$(\sigma_i,\ \sigma_c,\ \beta) \geq (\sigma_i',\ \sigma_c',\ \beta') = \mathtt{wf}\ \sigma_i \wedge \mathtt{wf}\ \sigma_c \wedge \sigma_i \sqsupseteq \sigma_i' \wedge \sigma_c \sqsupseteq \sigma_c' \wedge \mathtt{dom}(\beta) \backslash \mathtt{dom}(\beta') \subseteq \mathtt{dom}(\sigma_i) \backslash \mathtt{dom}(\sigma_i') \wedge \mathtt{dom}'(\beta) \backslash \mathtt{dom}'(\beta') \subseteq \mathtt{dom}(\sigma_c) \setminus \mathtt{dom}(\sigma_c')$
$(\sigma_i,\ \sigma_c,\ \beta,\ j) > (\sigma_i',\ \sigma_c',\ \beta',\ j') = (\sigma_i,\ \sigma_c,\ \beta) \geq (\sigma_i',\ \sigma_c',\ \beta')\ \wedge\ j < j'$
$(\sigma_i,\ \sigma_c,\ \beta,\ j) \geq (\sigma_i',\ \sigma_c',\ \beta',\ j') = (\sigma_i,\ \sigma_c,\ \beta) \geq (\sigma_i',\ \sigma_c',\ \beta')\ \wedge\ j \leq j'$

$$
\begin{array}{ll}
\mathcal{V}[\![(A)^{\mathbb{S}}]\!] & = \{\ (v_i,\ v_c,\ l,\ (\sigma,\ \sigma',\ \beta,\ k)) \mid l \notin \mathtt{dom}(\beta)\ \wedge\ (v_i,\ v_c,\ \sigma(l),\ (\sigma,\ \sigma',\ \beta,\ k)) \in \mathcal{V}_A[\![A]\!]\ \} \\
\mathcal{V}[\![\Box(\tau)]\!] & = \{\ (v_i,\ v_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ k)) \mid \mathcal{R}_{\sigma_i}(\{l\}) \cap \mathtt{dom}(\beta) = \emptyset\ \wedge\ (v_i,\ v_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ k)) \in \mathcal{V}[\![\tau]\!]\ \} \\
\mathcal{V}[\![(A)^{\mathbb{C}}]\!] & = \{\ (v_i,\ v_c,\ l,\ (\sigma,\ \sigma',\ \beta,\ k)) \mid \\
 & \quad l \notin \mathtt{dom}(\beta) \Rightarrow (v_i,\ v_c,\ \sigma(l),\ (\sigma,\ \sigma',\ \beta,\ k)) \in \mathcal{V}_A[\![A]\!]\ \wedge \\
 & \quad l \in \mathtt{dom}(\beta) \Rightarrow (\forall\ m,\ (v_i,\ l,\ (\sigma,\ \emptyset,\ m)) \in \mathcal{V}_A(\!|A|\!)\ \wedge\ (v_c,\ \beta(l),\ (\sigma \uplus \sigma',\ \beta,\ m)) \in \mathcal{V}_A(\!|A|\!))\ \} \\
\mathcal{V}[\![(A)^{\mathbb{C}}]\!] & = \mathcal{V}(\!|(A)^{\mathbb{C}}|\!)
\end{array}
$$

$$
\begin{array}{ll}
\mathcal{V}_A[\![\mathtt{unit}]\!] & = \{\ ((),\ (),\ (),\ W) \mid \top\ \} \\
\mathcal{V}_A[\![\tau_1 \times \tau_2]\!] & = \{\ ((v_{i1}, v_{i2}),\ (v_{c1}, v_{c2}),\ (v_{t1}, v_{t2}),\ W) \mid (v_{i1},\ v_{c1},\ v_{t1},\ W) \in \mathcal{V}[\![\tau_1]\!]\ \wedge\ (v_{i2},\ v_{c2},\ v_{t2},\ W) \in \mathcal{V}[\![\tau_2]\!]\ \} \\
\mathcal{V}_A[\![\mathtt{list}\,[n]^\alpha\ \tau]\!] & = \{\ (\mathtt{nil},\ \mathtt{nil},\ \mathtt{nil},\ W) \mid \models n \doteq 0 \wedge 0 \doteq \alpha\ \} \\
\mathcal{V}_A[\![\mathtt{list}\,[n]^\alpha\ \tau]\!] & = \{\ (\mathtt{cons}(v_i, vs_i),\ \mathtt{cons}(v_c, vs_c),\ \mathtt{cons}(\mathtt{inl}\ v_t, vs_t),\ W) \mid \\
 & \quad (v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\Box(\tau)]\!]\ \wedge (vs_i,\ vs_c,\ vs_t,\ W) \in \mathcal{V}[\![\mathtt{list}\,[n-1]^\alpha\ \tau]\!] \wedge \models 0 < n\ \} \cup \\
 & \quad \{\ (\mathtt{cons}(v_i, vs_i),\ \mathtt{cons}(v_c, vs_c),\ \mathtt{cons}(\mathtt{inl}\ v_t, vs_t),\ W) \mid \\
 & \quad (v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau]\!]\ \wedge (vs_i,\ vs_c,\ vs_t,\ W) \in \mathcal{V}[\![\mathtt{list}\,[n-1]^{\alpha-1}\ \tau]\!] \wedge \models 0 < n \wedge 0 < \alpha\ \} \cup \\
 & \quad \{\ (\mathtt{cons}(v_i, vs_i),\ \mathtt{cons}(v_c, vs_c),\ \mathtt{cons}(\mathtt{inr}\ v_t, vs_t),\ W) \mid \\
 & \quad (v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau]\!]\ \wedge (vs_i,\ vs_c,\ vs_t,\ W) \in \mathcal{V}[\![\mathtt{list}\,[n-1]^{\alpha-1}\ \tau]\!] \wedge \models 0 < n \wedge 0 < \alpha\ \} \\
\mathcal{V}_A[\![\exists t{::}S.\ \tau]\!] & = \{\ (\mathtt{pack}\ v_i,\ \mathtt{pack}\ v_c,\ v_t,\ W) \mid \exists I,\ \vdash I :: S\ \wedge\ (v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau[I/t]]\!]\ \} \\
\mathcal{V}_A(\!|C \supset \tau|\!) & = \{\ (v_i,\ v_c,\ v_t,\ W) \mid \neg \models C\ \vee\ (v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau]\!]\ \} \\
\mathcal{V}_A(\!|C\ \&\ \tau|\!) & = \{\ (v_i,\ v_c,\ v_t,\ W) \mid \models C\ \wedge\ (v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau]\!]\ \} \\
\mathcal{V}_A[\![\mathtt{int}]\!] & = \{\ (n,\ n,\ n,\ W) \mid \top\ \} \\
\mathcal{V}_A[\![\tau_1 + \tau_2]\!] & = \{\ (\mathtt{inl}\ v_i,\ \mathtt{inl}\ v_c,\ \mathtt{inl}\ v_t,\ W) \mid (v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau_1]\!]\ \} \cup \\
 & \quad \{\ (\mathtt{inr}\ v_i,\ \mathtt{inr}\ v_c,\ \mathtt{inr}\ v_t,\ W) \mid (v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau_2]\!]\ \} \\
\mathcal{V}_A[\![\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \tau_2]\!] & = \{\ (\mathtt{fix}\ f(x).\, e_i,\ \mathtt{fix}\ f(x).\, e_c,\ \mathtt{fix}\ f(x).\, e_t,\ W) \mid \\
 & \quad \forall v_i,\ v_c,\ v_t\ l\ (\sigma_i,\ \sigma_c,\ \beta,\ k) > W, (v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\tau_1]\!]\ \wedge\ \sigma_i(l) = \mathtt{fix}\ f(x).\, e_t \Rightarrow \\
 & \quad\quad ([x \mapsto v_i, f \mapsto \mathtt{fix}\ f(x).\, e_i]e_i,\ [x \mapsto v_c, f \mapsto \mathtt{fix}\ f(x).\, e_c]e_c,\ [x \mapsto v_t, f \mapsto l]e_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{E}[\![\tau_2]\!]^\kappa\ \} \\
\mathcal{V}_A[\![\tau_1 \xrightarrow{\mathbb{C}(\kappa)} \tau_2]\!] & = \mathcal{V}_A[\![\tau_1 \xrightarrow{\mathbb{S}(\kappa)} \tau_2]\!] \cap \\
 & \quad \{\ (\mathtt{fix}\ f(x).\, e_i,\ \mathtt{fix}\ f(x).\, e_c,\ \mathtt{fix}\ f(x).\, e_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ k)) \mid \\
 & \quad \forall m,\ (\mathtt{fix}\ f(x).\, e_i,\ \mathtt{fix}\ f(x).\, e_t,\ (\sigma_i,\ \emptyset,\ k)) \in \mathcal{V}_A(\!|\tau_1 \xrightarrow{\mathbb{C}(\kappa)} \tau_2|\!)\ \wedge \\
 & \quad\quad (\mathtt{fix}\ f(x).\, e_c,\ \mathtt{fix}\ f(x).\, e_t,\ (\sigma_i \uplus \sigma_c,\ \beta,\ m)) \in \mathcal{V}_A(\!|\tau_1 \xrightarrow{\mathbb{C}(\kappa)} \tau_2|\!)\ \} \\
\mathcal{V}_A[\![\forall t \overset{\mathbb{S}(\kappa)}{::} S.\ \tau]\!] & = \{\ (\Lambda.\, e_i,\ \Lambda.\, e_c,\ \Lambda.\, e_t,\ W) \mid \forall I,\ \vdash I :: S \Rightarrow (e_i,\ e_c,\ e_t,\ W) \in \mathcal{E}[\![\tau[I/t]]\!]^{\kappa[I/t]}\ \} \\
\mathcal{V}_A[\![\forall t \overset{\mathbb{C}(\kappa)}{::} S.\ \tau]\!] & = \mathcal{V}_A(\!|\forall t \overset{\mathbb{C}(\kappa)}{::} S.\ \tau|\!)
\end{array}
$$

$$
\begin{array}{ll}
\mathcal{E}[\![\tau]\!]^\kappa & = \{\ (e_i,\ e_c,\ e_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \mid \\
 & \quad \forall v_i\ j\ \sigma_i'\ \sigma_c'\ \beta'\ \sigma_o\ \beta_o\ c'\ t_1\ r, (\sigma_i',\ \sigma_c',\ \beta',\ j) \geq (\sigma_i,\ \sigma_c,\ \beta,\ m)\ \wedge\ \mathtt{stamps}(\mathtt{edges}(\sigma_i')) < t_1\ \wedge\ \mathtt{inv}(\sigma_i',\ \beta_o,\ e_t) \\
 & \quad\quad e_i \Downarrow v_i, j\ \wedge\ \mathcal{D}(\mathtt{edges}(\sigma_i'),\ \mathtt{dom}(\beta_o)),\ \sigma_i',\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c',\ \beta',\ c'\ \wedge \\
 & \quad\quad (\forall l_n,\ r = \mathbb{C}(l_n) \Rightarrow \sigma_i'(l_n) = \Box\ \wedge\ l_n \notin \mathtt{dom}(\beta')) \Rightarrow \\
 & \quad\quad \exists v_c\ v_t\ \ (\sigma_f,\ \sigma_f',\ \beta'') \geq (\sigma_i',\ \sigma_c',\ \beta')\ t_2\ c, \\
 & \quad\quad (1).\quad e_c \Downarrow v_c\ \wedge \\
 & \quad\quad (2).\quad e_t,\ \sigma_i',\ t_1 \Downarrow^r_{L_1,\emptyset} v_t,\ \sigma_f,\ t_2,\ \_\ \wedge\ (r = \mathbb{S} \Rightarrow \mathtt{inv}(\sigma_f,\ \beta_o,\ v_t))\ \wedge\ (r = \mathbb{C}(l) \Rightarrow \mathtt{inv}(\sigma_f[l \mapsto v_t],\ \beta_o,\ l)) \\
 & \quad\quad (3).\quad \mathcal{D}(\mathtt{edges}(\sigma_f),\ \mathtt{dom}(\beta_o)),\ \sigma_f,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_f',\ \beta'',\ c\ \wedge \\
 & \quad\quad (4).\quad c - c' \leq \kappa\ \wedge \\
 & \quad\quad (5).\quad r = \mathbb{S} \Rightarrow (v_i,\ v_c,\ v_t,\ (\sigma_f,\ \sigma_f',\ \beta'',\ m - j)) \in \mathcal{V}[\![\tau]\!] \\
 & \quad\quad (6).\quad \forall l_n,\ r = \mathbb{C}(l_n) \Rightarrow (v_i,\ v_c,\ l_n,\ (\sigma_f[l_n \mapsto v_t],\ \sigma_f',\ \beta'',\ m - j)) \in \mathcal{V}[\![\tau]\!]\ \}
\end{array}
$$

Figure 31: Binary step-indexed interpretation of types (Concrete semantics)

Proof. This fact has been proved using the Coq proof assistant. □

Lemma 26 (Dependency graph union (I))
If $\neg\texttt{path}(R,\ S_1 \cup S_2,\ l_d)$ and $\texttt{trg}(S_2) = \{l_d\}$ then $\mathcal{D}(S_1 \cup S_2,\ R) = \mathcal{D}(S_1,\ R)$.

Proof. Since the two lists have unique elements and they are sorted it suffices to show that the two lists have exactly the same sets of elements. This fact has been proved using the Coq proof assistant. □

Lemma 27 (Dependency graph union (II))
Assume that the following hold for the graphs S_1, S_2 and $\{(l_s,\ l_d,\ f,\ t_i,\ t_e)\}$.

1. $\texttt{path}(S_1,\ R,\ l_s)$
2. For all $(l_s,\ l_d,\ f,\ t_1,\ t_2) \in S_2$, $t_i < t_1 < t_2 < t_e$
3. For all $(l_s,\ l_d,\ f,\ t_1,\ t_2) \in S_1$, $t_1 < t_2 < t_i < t_e$,
4. For all $(l_s,\ l_d,\ f,\ t_1,\ t_2) \in S_1$, $(l'_s,\ l'_d,\ f',\ t'_1,\ t'_2) \in S_2$, $t'_1 < t'_2 < t_1 < t_2$
5. $\texttt{locs}(S_1) \cap \texttt{trg}(S_2) = \emptyset$

Then $\mathcal{D}(S_1 \cup S_2 \cup \{(l_s,\ l_d,\ f,\ t_i,\ t_e)\},\ R) = \mathcal{D}(S_1,\ R) +\!\!+ \{(l_s,\ l_d,\ f,\ t_i,\ t_e)\}$

Proof. Since the lists have unique elements, they are sorted and for all $(l_s,\ l_d,\ f,\ t_i,\ t_2) \in S_1$, $(l'_s,\ l'_d,\ f',\ t'_1,\ t'_2) \in S_2$, $t_1 < t_2 < t'_1 < t'_2$, it suffices to show that the two lists have exactly the same sets of elements. This fact has been formalized using the Coq proof assistant. □

Lemma 28 (Determinism of evaluation)
If

$$e,\ \sigma,\ t_1 \Downarrow^r_{L,\beta} v,\ \sigma',\ t_2,\ c$$

and

$$e,\ \sigma,\ t_1 \Downarrow^r_{L,\beta} v',\ \sigma'',\ t'_2,\ c'$$

then $v' = v$, $\sigma'' = \sigma'$, $t_2 = t'_2$ and $c' = c$.

Proof. By induction on the evaluation derivation. □

Lemma 29 (Evaluation invariants)
Let $\texttt{wf}\ \sigma$ and

$$e,\ \sigma,\ t_1 \Downarrow^r_{L,\beta} v,\ \sigma',\ t_2,\ c$$
$$\sigma = \sigma_1 \uplus \sigma_2$$
$$\sigma' = \sigma'_1 \uplus \sigma'_2$$

then the following hold:

1. $\sigma'_1 \sqsupseteq \sigma_1$ and $\sigma'_2 \sqsupseteq \sigma_2$
2. if $L = L_1$ then $\sigma'_2 = \sigma_2$
3. if $L = L_2$ then $\sigma'_1 = \sigma_1$
4. if $L = L_1$ then for all $\sigma''_2 \sqsupseteq \sigma_2$, $e,\ \sigma_1 \uplus \sigma''_2,\ t_1 \Downarrow^r_{L,\beta} v,\ \sigma'_1 \uplus \sigma''_2,\ t_2,\ c$
5. if $L = L_2$ then for all $\sigma''_1 \sqsupseteq \sigma_1$, $e,\ \sigma''_1 \uplus \sigma_2,\ t_1 \Downarrow^r_{L,\beta} v,\ \sigma''_1 \uplus \sigma'_2,\ t_2,\ c$

6. if $\texttt{FL}(e) \subseteq \texttt{dom}(\sigma)$ then $\texttt{wf}\ \sigma'$ and $\texttt{FL}(v) \subseteq \texttt{dom}(\sigma')$
7. if $\texttt{FL}(e) \subseteq \texttt{dom}(\sigma)$ then $\mathcal{R}_{\sigma'}(\texttt{FL}(v)) \subseteq \mathcal{R}_{\sigma}(\texttt{FL}(e)) \cup (\texttt{dom}(\sigma') \setminus \texttt{dom}(\sigma))$
8. if $\texttt{FL}(e) \subseteq \texttt{dom}(\sigma)$ then $\texttt{trg}(\texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma)) \subseteq (\texttt{dom}(\sigma') \setminus \texttt{dom}(\sigma)) \cup (r = \mathbb{C}(l)\ ?\ \{l\}\ :\ \emptyset)$
9. if $\texttt{FL}(e) \subseteq \texttt{dom}(\sigma)$ then $\texttt{src}(\texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma)) \subseteq \mathcal{R}_{\sigma}(\texttt{FL}(e)) \cup (\texttt{dom}(\sigma') \setminus \texttt{dom}(\sigma))$
10. if $\texttt{FL}(e) \subseteq \texttt{dom}(\sigma)$ then $\texttt{FL}(\texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma)) \subseteq \mathcal{R}_{\sigma}(\texttt{FL}(e)) \cup (\texttt{dom}(\sigma') \setminus \texttt{dom}(\sigma))$
11. if $\sigma(l) = \square$ then $l \notin \texttt{src}(\texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma))$
12. $\sigma'(l) = \square$ iff $\sigma(l) = \square$
13. For all $(l_s,\ l_d,\ f,\ t_i,\ t_e) \in \texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma)$, $t_1 \leq t_i < t_e < t_2$
14. For all $(l_s,\ l_d,\ f,\ t_1,\ t_2) \in \texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma)$, $(l'_s,\ l_d,\ f',\ t'_1,\ t'_2) \in \texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma)$ then $t'_1 < t_1 < t_2 < t'_2$ or $t_1 < t'_1 < t'_2 < t_2$
15. For all $(l_s,\ l_d,\ f,\ t_1,\ t_2) \in \texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma)$, $(l'_s,\ l'_d,\ f',\ t'_1,\ t'_2) \in \texttt{edges}(\sigma') \setminus \texttt{edges}(\sigma)$ then $t'_1 \neq t_1$ and $t_2 \neq t'_2$

Proof. By induction on the evaluation derivation. □

Lemma 30 (Change propagation invariants)
If

$$D,\ \sigma_f,\ \sigma_c,\ \beta \rightsquigarrow \sigma'_f,\ \beta,\ c$$

then the following hold:

1. $\sigma'_f \sqsupseteq \sigma_c$
2. $\beta' \sqsupseteq \beta$
3. if $\texttt{wf}\ \sigma_c$, $\texttt{wf}\ \sigma_f$, and $\texttt{FL}(D) \subseteq \sigma_f$ then $\texttt{wf}\ \sigma'_f$
4. $l \in \texttt{trg}(D)$ if and only if $l \in \texttt{dom}(\beta') \setminus \texttt{dom}(\beta)$

Proof. The result follows by induction on change propagation derivation using Lemma 29. □

Lemma 31 (Change propagation is deterministic)
If

$$D,\ \sigma_c,\ \sigma_f,\ \beta \rightsquigarrow \sigma'_f,\ \beta',\ c$$

and

$$D,\ \sigma_c,\ \sigma_f,\ \beta \rightsquigarrow \sigma''_f,\ \beta'',\ c'$$

then $\sigma''_f = \sigma'_f$, $\beta'' = \beta'$ and $c = c'$.

Proof. The result follows by induction on change propagation derivation using Lemma 28. □

Lemma 32 (Change propagation under store extension)
If

$$D,\ \sigma_c,\ \sigma_f,\ \beta \rightsquigarrow \sigma'_f,\ \beta',\ c$$

and then for all $\sigma''_f \sqsupseteq \sigma_f$

$$D,\ \sigma_c,\ \sigma''_f,\ \beta \rightsquigarrow \sigma'_f,\ \beta',\ c$$

Proof. The result follows by induction on change propagation derivation using Lemma 29. □

Lemma 33 (Change propagation composition)
If

$$D_1,\ \sigma_c,\ \sigma_f,\ \beta \rightsquigarrow \sigma'_f,\ \beta',\ c_1 \tag{1}$$
$$D_2,\ \sigma'_f,\ \sigma_f,\ \beta' \rightsquigarrow \sigma''_f,\ \beta'',\ c_2$$

then

$$D_1 +\!\!+ D_2,\ \sigma_c,\ \sigma_f,\ \beta \rightsquigarrow \sigma''_f,\ \beta'',\ c_1 + c_2$$

Proof. The result follows by induction on the change propagation derivation of eq. (1). □

Lemma 34 (No free locations)
Assume that

$$\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner$$

Then $\mathtt{FL}(\ulcorner e \urcorner) = \emptyset$.

Proof. It follows induction on the typing derivation. □

Theorem 35 (Translation is type preserving)
If $\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner$ then $\|\Gamma\| \vdash \ulcorner e \urcorner : \|\tau\| \mid$

Proof. It follows by induction on the typing derivation. □

Lemma 36 (World extension closure)
The following hold:

1. If $(v_s,\ v_t,\ W) \in \mathcal{V}_A(\!|A|\!)$ and $W' \geq W$, then $(v_s,\ v_t,\ W') \in \mathcal{V}_A(\!|A|\!)$.
2. If $(v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau|\!)$ and $W' \geq W$, then $(v_s,\ v_t,\ W') \in \mathcal{V}(\!|\tau|\!)$.
3. If $(e_s,\ e_t,\ W) \in \mathcal{E}(\!|\tau|\!)^\kappa$ and $W' \geq W$, then $(e_s,\ e_t,\ W') \in \mathcal{E}(\!|\tau|\!)^\kappa$.
4. If $(\theta_s,\ \theta_t,\ W) \in \mathcal{G}(\!|\Gamma|\!)$ and $W' \geq W$, then $(\theta_s,\ \theta_t,\ W') \in \mathcal{G}(\!|\Gamma|\!)$.
5. If $(v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}_A[\![A]\!]$ and $W' \geq W$, then $(v_i,\ v_c,\ v_t,\ W') \in \mathcal{V}_A[\![A]\!]$.
6. If $(v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau]\!]$ and $W' \geq W$, then $(v_i,\ v_c,\ v_t,\ W') \in \mathcal{V}[\![\tau]\!]$.
7. If $(e_i,\ e_c,\ e_t,\ W) \in \mathcal{E}[\![\tau]\!]^\kappa$ and $W' \geq W$, then $(e_i,\ e_c,\ e_t,\ W') \in \mathcal{E}[\![\tau]\!]^\kappa$.
8. If $(\theta_i,\ \theta_c,\ \theta_t,\ W) \in \mathcal{G}[\![\Gamma]\!]$ and $W' \geq W$, then $(\theta_i,\ \theta_c,\ \theta_t,\ W') \in \mathcal{G}[\![\Gamma]\!]$.

Proof. First we prove statements (1), (2) and (3) simultaneously. Statements (1) and (2) are proved by mutual induction on A and τ. Similarly, we prove statements (5), (6) and (7).

Finally, we can prove statement (4) (resp. (8)) by induction on the length of the environment and using statement (2) (resp. (6)). □

Lemma 37 (Value relation projection)
The following hold.

1. If $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ j)) \in \mathcal{V}[\![\tau]\!]$ then $\forall\, m,\ (v_i,\ v_t,\ (\sigma,\ \emptyset,\ j)) \in \mathcal{V}(\!|\tau|\!)$ and $\forall\, m,\ (v_c,\ v_t,\ (\sigma_i \uplus \sigma_c,\ \beta,\ m)) \in \mathcal{V}(\!|\tau|\!)$
2. If $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ j)) \in \mathcal{V}_A[\![A]\!]$ then $\forall\ m,\ (v_i,\ v_t,\ (\sigma,\ \emptyset,\ j)) \in \mathcal{V}_A(\!|A|\!)$ and $\forall\ m,\ (v_c,\ v_t,\ (\sigma_i \uplus \sigma_c,\ \beta,\ m)) \in \mathcal{V}_A(\!|A|\!)$

Proof. We prove the two statements simultaneously by mutual induction on τ and A. □

Lemma 38 (Value relation injection)
The following hold.

1. If
$$\forall\ m,\ (v_i,\ l,\ (\sigma_i,\ \emptyset,\ j)) \in \mathcal{V}(\!|\tau|\!)$$
$$\forall\ m,\ (v_c,\ l,\ (\sigma_i \uplus \sigma_c,\ \beta,\ m)) \in \mathcal{V}(\!|\tau|\!)$$
and
$$\models \mathbb{C} \trianglelefteq \tau$$
then $(v_i,\ v_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}(\!|\tau|\!)$

Proof. It follows by case analysis on the type τ and the definition of the value relation. □

Lemma 39 (Stable type lemma)
The following hold.

1. $(e_s,\ e_t,\ W) \in \mathcal{E}(\!|\tau|\!)^\kappa$ if and only if $(e_s,\ e_t,\ W) \in \mathcal{E}(\!|\Box(\tau)|\!)^\kappa$
2. If $\mathcal{R}_{\sigma_i}(\{l\}) \cap \mathtt{dom}(\beta) = \emptyset$ and $(e_i,\ e_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{E}[\![\tau]\!]^\kappa$ then $(e_i,\ e_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{E}[\![\Box(\tau)]\!]^0$

Proof of statement 1. The result follows easily by the definition of the expression relation. Note that $\mathcal{V}(\!|\tau|\!) = \mathcal{V}(\!|\Box(\tau)|\!)$. □

Proof of statement 2. Let $(e_i,\ e_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{E}[\![\tau]\!]^\kappa$ and $\mathcal{R}_{\sigma_i}(\{l\}) \cap \mathtt{dom}(\beta) = \emptyset$. We pick arbitrary v_i, j_i, σ_i, σ_c, β, σ_o, β_o, c', t_0 and r such that

$$\mathcal{D}(\mathtt{edges}(\sigma_i),\ \mathtt{dom}(\beta_o)),\ \sigma_i,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c' \tag{1}$$

$$(\sigma_i,\ \sigma_c,\ \beta,\ j_i) \geq W$$
$$\theta_i(e_1\ e_2) \Downarrow v_i \tag{2}$$

and
$$r = \mathbb{C}(l) \Rightarrow \sigma_i(l) = \Box\ \wedge\ l \notin \mathtt{dom}(\beta)$$

We instantiate the hypothesis with $v)i$, j_1, σ_i, σ_c, β, (note that $(\sigma_i,\ \sigma_c,\ \beta,\ j_1) \geq W$), σ_o, β_o, c', $\mathbb{S}$, t_0, eq. (1) and eq. (2) and we obtain v_c, v_t,$(\sigma_1,\ \sigma_1',\ \beta_1) \geq (\sigma_i,\ \sigma_c,\ \beta)$, t_1, c_1 such that:

$$e_c \Downarrow v_c \tag{A1}$$

$$e_c,\ \sigma_i,\ t_0 \Downarrow^r_{L,\beta} v_t,\ \sigma_1,\ t_1,\ \_ \tag{A2}$$

$$(r = \mathbb{S} \Rightarrow \mathtt{inv}(\sigma_1,\ \beta_o,\ v_t))\ \wedge\ (r = \mathbb{C}(l) \Rightarrow \mathtt{inv}(\sigma_1[l \mapsto v_t],\ \beta_o,\ l)) \tag{A3}$$

$$\mathcal{D}(\mathtt{edges}(\sigma_i),\ \sigma_1)\mathtt{dom}(\beta),\ \sigma_c,\ \sigma_1,\ \beta \rightsquigarrow \sigma_1',\ \beta_1,\ c_1 \tag{A4}$$

$$c_1 \leq \kappa \tag{A5}$$

$$r = \mathbb{S} \Rightarrow (v_i,\ v_c,\ v_t,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_i)) \in \mathcal{V}[\![\tau]\!] \tag{A6}$$

and
$$r = \mathbb{C}(l) \Rightarrow (v_i,\ v_c,\ l,\ (\sigma_1[l \mapsto v_t],\ \sigma_1',\ \beta_1,\ m - j_i)) \in \mathcal{V}[\![\tau]\!] \tag{A7}$$

Goals 1-2 result from eq. (A1), eq. (A2) and eq. (A3). From Lemma 29 we derive that $\mathtt{locs}(\mathtt{edges}(\sigma_1) \setminus \mathtt{edges}(\sigma_i)) \subset (\mathtt{dom}(\sigma_1) \setminus \mathtt{dom}(\sigma_i)) \cup \mathcal{R}_{\sigma_i}(\mathtt{FL}(e_t)) \cup ((m = \mathbb{C}(l)\ ?\ \{l\}\ :\ \emptyset))$

We know that $l \notin \mathtt{dom}(\beta)$, $\mathtt{dom}(\beta) \subseteq \mathtt{dom}(\sigma_i)$ and $\mathcal{R}_{\sigma_i}(\mathtt{FL}(e_t)) \cap \mathtt{dom}(\beta) = \emptyset$. From this we can easily derive that $\mathcal{D}(\mathtt{edges}(\sigma_1),\ \mathtt{dom}(\beta_o)) = \mathcal{D}(\mathtt{edges}(\sigma_i),\ \mathtt{dom}(\beta_o))$. Thus we derive goal 3

$$\mathcal{D}(\mathtt{edges}(\sigma_1),\ \mathtt{dom}(\beta_o)),\ \sigma_i,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c' \tag{A3}$$

For goal 4 we trivially derive that $c_i - c_i \leq 0$. For goal 5 we assume that $r = \mathbb{S}$. From eq. (A6) we know that

$$(v_i,\ v_c,\ v_t,\ (\sigma_1,\ \sigma_c,\ \beta,\ m - j_i)) \in \mathcal{V}[\![\tau]\!]$$

Using lemma Lemma 29 we derive that $\mathcal{R}_{\sigma_1}(\mathtt{FL}(v_t)) \subseteq (\sigma_1 \backslash \sigma_i) \cup \mathcal{R}_{\sigma_i}(e_t)$. Since $(\sigma_1 \backslash \sigma_i) \cap \mathtt{dom}(\beta) = \emptyset$ and $\mathcal{R}_{\sigma_i}(e_t) \cap \mathtt{dom}(\beta) = \emptyset$ we derive that $\mathcal{R}_{\sigma_1}(\mathtt{FL}(v_t)) \cap \mathtt{dom}(\beta) = \emptyset$ which proves that

$$(v_i,\ v_c,\ v_t,\ (\sigma_1,\ \sigma_c,\ \beta,\ m - j_i)) \in \mathcal{V}[\![\Box(\tau)]\!]$$

For goal 6 we assume that $r = \mathbb{C}(l)$. As above we can derive that

$$(v_i,\ v_c,\ l,\ (\sigma_1[l \mapsto v_t],\ \sigma_1',\ \beta,\ m - j_i)) \in \mathcal{V}[\![\Box(\tau)]\!]$$

□

Lemma 40 (Value interpretation containment)
The following hold:

1. If $(v_s,\ v_t,\ W) \in \mathcal{V}_A(\!|A|\!)$ then $(v_s,\ \mathtt{ref}\ v_t,\ W) \in \mathcal{E}(\!|(A)^\mu|\!)^1$.
2. If $(v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}_A[\![A]\!]$ then $(v_i,\ v_c,\ \mathtt{ref}\ v_t,\ W) \in \mathcal{E}[\![(A)^\mu]\!]^0$.
3. If $(v_s,\ v_t,\ W) \in \mathcal{V}(\!|\tau|\!)$ then $(v_s,\ v_t,\ W) \in \mathcal{E}(\!|\tau|\!)^0$.
4. If $(v_i,\ v_c,\ v_t,\ W) \in \mathcal{V}[\![\tau]\!]$ then $(v_i,\ v_c,\ v_t,\ W) \in \mathcal{E}[\![\tau]\!]^0$.

Proof. All of the statements follow easily from the definition of the expression relation. □

Lemma 41 (Unary interpretation unfolding)
Assume that:

$$\forall\ m,\ (e_c,\ e_t,\ (\sigma,\ \beta))m \in \mathcal{E}(\!|\tau|\!)^\kappa \tag{A}$$

Then for all t_1, r, such that $r = \mathbb{C}(l) \rightarrow \sigma(l) = \Box$, there exist v_c, v_t, σ':

1. $e_i \Downarrow v_i$
2. $e_c,\ \sigma,\ t_1 \Downarrow^r_{L,\beta} v_t,\ \sigma',\ t_2,\ cl$
3. $\models c \leq \kappa$
4. $r = \mathbb{S} \Rightarrow\ \forall\ m,\ (v_i,\ v_t,\ (\sigma',\ \beta,\ m)) \in \mathcal{V}(\!|\tau|\!)$
5. $r = \mathbb{C}(l) \Rightarrow\ \forall\ m,\ (v_i,\ l,\ (\sigma'[l \mapsto v_t],\ \beta,\ m)) \in \mathcal{V}(\!|\tau|\!)$

Proof. We instantiate eq. (A) with step index $\kappa + 1$ and obtain v_c, σ', t_2, c such that

$$e_c \Downarrow v_c$$

$$e_c,\ \sigma,\ t_1 \Downarrow^r_{L,\beta} v_t,\ \sigma',\ t_2,\ c$$

and

$$\models c \leq \kappa$$

Assume that $r = \mathbb{S}$. We pick an arbitrary m. We instantiate eq. (A) with $m + \kappa + 1$. Using the fact that the evaluation relation is deterministic in the source and $\kappa < m + \kappa + 1$ we can obtain v_t', σ'', t_2' and c such that

$$e_c,\ \sigma,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} v_t',\ \sigma'',\ t_2',\ c$$

and

$$(v_i,\ v_t,\ (\sigma'',\ \beta,\ m + \kappa + 1 - c)) \in \mathcal{V}(\!|\tau|\!)$$

Using lemma Lemma 28 we can we can show that $v_t' = v_t$ and $\sigma'' = \sigma'$, thus

$$(v_i,\ v_t,\ (\sigma',\ \beta,\ m + \kappa + 1 - c)) \in \mathcal{V}(\!|\tau|\!)$$

Using Lemma 36 and the fact that $\models c \leq \kappa$ we can show that

$$(v_i,\ v_t,\ (\sigma',\ m)) \in \mathcal{V}(\!|\tau|\!)$$

Finally, assume that $r = \mathbb{C}(l)$. We pick an arbitrary m. We instantiate eq. (A) with $m+\kappa+1$. Using the fact that the evaluation relation is deterministic in the source and $\kappa < m + \kappa + 1$ we can obtain v_t', σ'', t_2' and c such that

$$e_c,\ \sigma,\ t_1 \Downarrow^{\mathbb{C}(l)}_{L,\beta} v_t',\ \sigma'',\ t_2',\ c$$

and

$$(v_i,\ l,\ (\sigma''[l \mapsto v_t'],\ \beta,\ m + \kappa + 1 - c)) \in \mathcal{V}(\!|\tau|\!)$$

Using lemma Lemma 28 we can we can show that $v_t' = v_t$ and $\sigma'' = \sigma'$, thus

$$(v_i,\ l,\ (\sigma'[l \mapsto v_t],\ \beta,\ m + \kappa + 1 - c)) \in \mathcal{V}(\!|\tau|\!)$$

Using Lemma 36 and the fact that $\models c \leq \kappa$ we can show that

$$(v_i,\ l,\ (\sigma'[l \mapsto v_t],\ m)) \in \mathcal{V}(\!|\tau|\!)$$

□

Theorem 42 (Subtyping Soundness - Unary interpretation)
The following hold.

1. If $\Delta; \Phi \models A' \sqsubseteq A$, $\varphi \in \mathcal{D}[\![\Delta]\!]$, $\models \varphi\Phi$ and $(v_s,\ v_t,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{V}_A(\!|\varphi A'|\!)$ then $(v_s,\ v_t,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{V}(\!|\varphi A|\!)$
2. If $\Delta; \Phi \models \tau' \sqsubseteq \tau$, $\varphi \in \mathcal{D}[\![\Delta]\!]$, $\models \varphi\Phi$ and $(v_s,\ v_t,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{V}(\!|\varphi\tau'|\!)$ then $(v_s,\ v_t,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{V}(\!|\varphi\tau|\!)$
3. If $\Delta; \Phi \models \tau' \sqsubseteq \tau$, $\varphi \in \mathcal{D}[\![\Delta]\!]$, $\models \varphi\Phi$ and $(e_s,\ e_t,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{E}(\!|\varphi\tau'|\!)^{\kappa}$ and $\phi\kappa \leq \phi\kappa'$ then $(e_s,\ e_t,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{E}(\!|\varphi\tau|\!)^{\kappa'}$

Proof. We prove the above statements simultaneously. Statement 3 follows by the definition of the expression relation and the inductive hypothesis of statement 2. Statements 1 and 2 are proved by induction on the subtyping derivation. The cases follow easily. Note that $\mathcal{V}(\!|\tau|\!) = \mathcal{V}(\!|\Box(\tau)|\!)$ and that for all α, β, $\mathcal{V}(\!|\texttt{list}\ [n]^{\alpha}\ \tau|\!) = \mathcal{V}(\!|\texttt{list}\ [n]^{\beta}\ \tau|\!)$. □

Theorem 43 (Subtyping Soundness - Binary interpretation)
The following hold.

1. If $\Delta;\Phi \models A' \sqsubseteq A$, $\varphi \in \mathcal{D}[\![\Delta]\!]$, $\models \varphi\Phi$ and $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}_A[\![\varphi A']\!]$ then $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}_A[\![\varphi A]\!]$
2. If $\Delta;\Phi \models \tau' \sqsubseteq \tau$, $\varphi \in \mathcal{D}[\![\Delta]\!]$, $\models \varphi\Phi$ and $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\varphi\tau']\!]$ then $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\varphi\tau]\!]$
3. If $\Delta;\Phi \models \tau' \sqsubseteq \tau$, $\varphi \in \mathcal{D}[\![\Delta]\!]$, $\models \varphi\Phi$ and $(e_i,\ e_c,\ e_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{E}[\![\varphi\tau']\!]^\kappa$ and $\phi\kappa \leq \phi\kappa'$ then $(e_i,\ e_c,\ e_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{E}[\![\varphi\tau]\!]^{\kappa'}$

Proof. We prove the above statements simultaneously.

Proof of statement 3. Let $\varphi \in \mathcal{D}[\![\Delta]\!]$, $\models \varphi\Phi$ and

$$(e_i,\ e_c,\ e_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{E}[\![\varphi A]\!]^\kappa \tag{A}$$

We pick arbitrary v_i, j_i, σ_i, σ_c, β, σ_o, β_o, c', t_0 and r such that

$$\mathcal{D}(\mathtt{edges}(\sigma_i),\ \mathtt{dom}(\beta_o)),\ \sigma_i,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c' \tag{1}$$

$$(\sigma_i,\ \sigma_c,\ \beta,\ j_i) \geq W$$

$$\theta_i(e_1\ e_2) \Downarrow v_i \tag{2}$$

and

$$r = \mathbb{C}(l) \Rightarrow \sigma_i(l) = \square\ \wedge\ l \notin \mathtt{dom}(\beta)$$

We instantiate the hypothesis with $v)i$, j_1, σ_i, σ_c, β, (note that $(\sigma_i,\ \sigma_c,\ \beta,\ j_1) \geq W$), σ_o, β_o, c', $\mathbb{S}$, t_0, eq. (1) and eq. (2) and we obtain v_c, v_t,$(\sigma_1,\ \sigma_1',\ \beta_1) \geq (\sigma_i,\ \sigma_c,\ \beta)$, t_1, c_1 such that:

$$e_c \Downarrow v_c \tag{A1}$$

$$e_t,\ \sigma_i,\ t_0 \Downarrow^r_{L,\beta} v_t,\ \sigma_1,\ t_1,\ \_ \tag{A2}$$

$$\mathcal{D}(\mathtt{edges}(\sigma_i),\ \sigma_1)\mathtt{dom}(\beta),\ \sigma_c,\ \sigma_1,\ \beta \rightsquigarrow \sigma_1',\ \beta_1,\ c_1 \tag{A4}$$

$$(r = \mathbb{S} \Rightarrow \mathtt{inv}(\sigma_1,\ \beta_o,\ v_t))\ \wedge\ (r = \mathbb{C}(l) \Rightarrow \mathtt{inv}(\sigma_1[l \mapsto v_t],\ \beta_o,\ l)) \tag{A3}$$

$$\mathcal{D}(\mathtt{edges}(\sigma_i),\ \sigma_1)\mathtt{dom}(\beta),\ \sigma_c,\ \sigma_1,\ \beta \rightsquigarrow \sigma_1',\ \beta_1,\ c \tag{A4}$$

$$c \leq \varphi\kappa \tag{A5}$$

$$m = \mathbb{S} \Rightarrow (v_i,\ v_c,\ v_t,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau']\!] \tag{A6}$$

and

$$m = \mathbb{C}(l) \Rightarrow (v_i,\ v_c,\ l,\ (\sigma_1[l \mapsto v_t],\ \sigma_1',\ \beta_1,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau']\!] \tag{A7}$$

Goals 1-3 follow by eq. (A1), eq. (A2), eq. (A3) and eq. (A4). We can use eq. (A5) and the fact that $\varphi\kappa \leq \varphi\kappa'$ to prove that $c_1 \leq \varphi\kappa'$. Finally from the induction hypothesis of statement 2 we derive that

$$m = \mathbb{S} \Rightarrow (v_i,\ v_c,\ v_t,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - c)) \in \mathcal{V}[\![\varphi\tau]\!]$$

and

$$m = \mathbb{C}(l) \Rightarrow (v_i,\ v_c,\ l,\ (\sigma_1[l \mapsto v_t],\ \sigma_1',\ \beta_1,\ m - c)) \in \mathcal{V}[\![\varphi\tau]\!]$$

□

Proof of statement 1. For each of the following cases we pick φ such that $\varphi \in \mathcal{D}[\![\Delta]\!]$ and $\models \varphi\Phi$.

Case $\dfrac{}{\Delta;\Phi \models^{\mathbf{A}} (\texttt{real})^{\mathbb{S}} \sqsubseteq \Box((\texttt{real})^{\mu})}$ **real**

Let

$$(r_1,\ r_2,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![(\texttt{real})^{\mathbb{S}}]\!]$$

From the definition of the relation we derive that $\sigma_i(l) = r$, $l \notin \beta$ and that $r_1 = r_2 = r$.

We need to show that

$$(r,\ r,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\Box((\texttt{real})^{\mu})]\!]$$

It suffices to show that

$$(r,\ r,\ r,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}_A[\![\texttt{real}]\!]$$

which follows from the definition of the relation, and that $\mathcal{R}_{\sigma_i}(\{l\}) \cap \texttt{dom}(\beta) = \emptyset$. The latter follows from the facts that $\mathcal{R}_{\sigma_i}(\{l\}) = \{l\}$ and $l \notin \texttt{dom}(\beta)$

Case $\dfrac{}{\Delta;\Phi \models \Box((\tau_1 \xrightarrow{\delta(\kappa)} \tau_2)^{\mu}) \sqsubseteq (\Box(\tau_1) \xrightarrow{\delta(\kappa)} \Box(\tau_2))^{\mathbb{S}}} \to \Box$

Let

$$(v_i,\ v_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\Box((\varphi\tau_1 \xrightarrow{\delta(\varphi\kappa)} \varphi\tau_2)^{\mu})]\!] \tag{A}$$

From this we derive that $\mathcal{R}_{\sigma_i}(\{l\}) \cap \texttt{dom}(\beta) = \emptyset$, $\sigma(l) = \texttt{fix}\ f(x).\, e_t\ v_i = \texttt{fix}\ f(x).\, e_i$, $v_c = \texttt{fix}\ f(x).\, e_c$ and $l \notin \texttt{dom}(\beta)$. To show the goal we pick arbitrary $W' = (\sigma_i',\ \sigma_c',\ \beta',\ m') > (\sigma_i,\ \sigma_c,\ \beta,\ m)$ and $(v_i',\ v_c',\ l',\ W') \in \mathcal{V}[\![\Box(\varphi\tau_1)]\!]$. From this we obtain that $\mathcal{R}_{\sigma_i'}(\{l'\}) \cap \texttt{dom}(\beta)' = \emptyset$ and $(v_i',\ v_c',\ l',\ W') \in \mathcal{V}[\![\varphi\tau_1]\!]$. We instantiate eq. (A) with $(v_i',\ v_c',\ l',\ W') \in \mathcal{V}[\![\Box(\varphi\tau_1)]\!]$ and we derive

$$(x \mapsto v_i', f \mapsto \texttt{fix}\ f(x).\, e_i]e_i,\ x \mapsto v_c', f \mapsto \texttt{fix}\ f(x).\, e_c]e_c,\ [x \mapsto l', f \mapsto \texttt{fix}\ f(x).\, e_c]e_t,\ W') \in \mathcal{E}[\![\varphi\tau_2]\!]^{\varphi\kappa}$$

We can easily show that $\mathcal{R}_{\sigma_i'}(\texttt{FL}([x \mapsto l', f \mapsto \texttt{fix}\ f(x).\, e_t]e_t)) \cap \texttt{dom}(\beta') = \emptyset$. From Lemma 39 and statement 3 for $\models 0 \leq \varphi\kappa$, we derive that

$$(x \mapsto v_i', f \mapsto \texttt{fix}\ f(x).\, e_i]e_i,\ x \mapsto v_c', f \mapsto \texttt{fix}\ f(x).\, e_c]e_c,\ [x \mapsto v_t', f \mapsto \texttt{fix}\ f(x).\, e_c]e_t,\ W') \in \mathcal{E}[\![\Box(\varphi\tau_2')]\!]^{\varphi\kappa}$$

which proves the goal.

Case $\dfrac{}{\Delta;\Phi \models \Box((\texttt{list}\,[n]^{\alpha}\ \tau)^{\mu}) \sqsubseteq (\texttt{list}\,[n]^{\alpha}\ \Box(\tau))^{\mathbb{S}}}$ **l**$\Box$

We will show that for all α, n if

$$(v_i,\ v_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\Box((\texttt{list}\,[n]^{\alpha}\ \tau)^{\mu})]\!]$$

then

$$(v_i,\ v_c,\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![(\texttt{list}\,[n]^{\alpha}\ \Box(\tau))^{\mathbb{S}}]\!]$$

From the premise we derive that $\sigma(l) = v_t$, $\mathcal{R}_{\sigma_i}(\{l\}) \cap \texttt{dom}(\beta) = \emptyset$, $l \notin \texttt{dom}(\beta)$ and

$$(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}_A[\![\texttt{list}\,[n]^{\alpha}\ \tau]\!]\mu \tag{A}$$

. We proceed by subinduction on v_i.

- $v_i = \texttt{nil}$
  From eq. (A) we derive that $v_c = \texttt{nil}$, $v_t = \texttt{nil}$ and $\models n = 0$ and $\models a = 0$. From the definition of the relation we show that

  $$(\texttt{nil},\ \texttt{nil},\ \texttt{nil},\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n]^{\alpha}\ \Box(\varphi\tau)]\!]$$

- $v_i = \texttt{cons}(v_i',\ vs_i)$
  From eq. (A) we derive that $v_c = \texttt{cons}(v_t',\ vs_t)$, $v_t = \texttt{cons}(\texttt{inr}\ v_t',\ vs_t)$ or $v_t = \texttt{cons}(\texttt{inr}\ v_t',\ vs_t)$
  We consider the two cases separately.
  - $v_t = \texttt{cons}(\texttt{inl}\ v_t',\ vs_t)$
    From eq. (A) we derive

    $$(v_i',\ v_c',\ v_t',\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\Box(\varphi\tau)]\!] \tag{B}$$

    and

    $$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^{\alpha}\ \varphi\tau]\!]$$

    $$\models n > 0$$

    or

    $$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^{\alpha-1}\ \varphi\tau]\!]$$

    $$\models n > 0\ \wedge\ \models a > 0$$

    In the first case we can use the sub-induction hypothesis to derive that

    $$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^{\alpha}\ \Box(\varphi\tau)]\!] \tag{C}$$

    We can combine eq. (B) and eq. (C) in order to derive

    $(\texttt{cons}(v_i',\ vs_i),\ \texttt{cons}(v_c',\ vs_c),\ \texttt{cons}(\texttt{inl}\ v_t',\ vs_t),\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n]^{\alpha}\ \Box(\varphi\tau)]\!]$

    The other case is similar.

– $v_t = \texttt{cons}(\texttt{inr}\ v'_t,\ vs_t)$

From eq. (A) we derive

$$(v'_i,\ v'_c,\ v'_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\varphi\tau]\!]$$

and

$$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^{\alpha-1}\ \Box(\varphi\tau)]\!]$$
$$\models n > 0\ \wedge\ \models a > 0$$

We can use the sub-induction hypothesis to derive that

$$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^{\alpha-1}\ \Box(\varphi\tau')]\!]$$

Also, we can show that $\mathcal{R}_{\sigma_i}(v'_t) \cap \texttt{dom}(\beta) = \emptyset$, and thus

$$(v'_i,\ v'_c,\ v'_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\Box(\varphi\tau)]\!]$$

We can combine the two statements above in order to derive

$$(\texttt{cons}(v'_i,\ vs_i),\ \texttt{cons}(v'_c,\ vs_c),\ \texttt{cons}(\texttt{inr}\ v'_t,\ vs_t),\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n]^{\alpha}\ \Box(\varphi\tau)]\!]$$

□

Proof of statement 2. For each of the following cases we pick φ such that $\varphi \in \mathcal{D}[\![\Delta]\!]$ and $\models \varphi\Phi$.

Case $\dfrac{\Delta;\Phi \models \tau'_1 \sqsubseteq \tau_1 \qquad \Delta;\Phi \models \tau_2 \sqsubseteq \tau'_2 \qquad \Delta;\Phi \models \kappa \leq \kappa'}{\Delta;\Phi \models^{\mathbf{A}} \tau_1 \xrightarrow{\delta(\kappa)} \tau_2 \sqsubseteq \tau'_1 \xrightarrow{\delta(\kappa')} \tau'_2} \to 1$

Let

$$(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\tau_1 \xrightarrow{\delta(\kappa)} \tau_2]\!] \tag{A}$$

From this we derive that $v_i = \texttt{fix}\ f(x).\,e_i$, $v_c = \texttt{fix}\ f(x).\,e_c$ and $v_t = \texttt{fix}\ f(x).\,e_t$. To show the goal we pick arbitrary $W' > (\sigma_i,\ \sigma_c,\ \beta,\ m)$ and $(v'_i,\ v'_c,\ v'_t,\ W') \in \mathcal{V}[\![\varphi\tau'_1]\!]$. From the induction hypothesis we derive that $(v'_i,\ v'_c,\ v'_t,\ W) \in \mathcal{V}[\![\varphi\tau'_2]\!]$. We instantiate eq. (A) with this fact and we derive

$$(x \mapsto v'_i, f \mapsto \texttt{fix}\ f(x).\,e_i]e_i,\ x \mapsto v'_c, f \mapsto \texttt{fix}\ f(x).\,e_c]e_c,\ [x \mapsto v'_t, f \mapsto \texttt{fix}\ f(x).\,e_c]e_t,\ W') \in \mathcal{E}[\![\varphi\tau_2]\!]^{\varphi\kappa}$$

From statement 3, we derive

$$(x \mapsto v'_i, f \mapsto \texttt{fix}\ f(x).\,e_i]e_i,\ x \mapsto v'_c, f \mapsto \texttt{fix}\ f(x).\,e_c]e_c,\ [x \mapsto v'_t, f \mapsto \texttt{fix}\ f(x).\,e_c]e_t,\ W') \in \mathcal{E}[\![\varphi\tau'_2]\!]^{\varphi\kappa'}$$

Case $\dfrac{\Delta; \Phi \models \alpha \doteq 0}{\Delta; \Phi \models^{\mathsf{A}} \mathtt{list}\,[n]^{\alpha}\ \tau \sqsubseteq \mathtt{list}\,[n]^{\alpha}\ \Box(\tau)}$ **l2***

Let

$$(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[\varphi n]^{\varphi\alpha}\ \varphi\tau]\!] \tag{A}$$

Note that from the premise $\models \varphi\alpha = 0$. We will prove that for all n, if $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[n]^{\varphi\alpha}\ \varphi\tau]\!]$ then $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[n]^{\varphi\alpha}\ \Box(\varphi\tau)]\!]$. We proceed by subinduction on v_i.

- $v_i = \mathtt{nil}$
  From eq. (A) we derive that $v_c = \mathtt{nil}$, $v_t = \mathtt{nil}$ and $\models \varphi n = 0$. From the definition of the relation we show that

  $$(\mathtt{nil},\ \mathtt{nil},\ \mathtt{nil},\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[n]^{\varphi\alpha}\ \Box(\varphi\tau)]\!]$$

- $v_i = \mathtt{cons}(v_i',\ vs_i)$
  From eq. (A) we derive that $v_c = \mathtt{cons}(v_t',\ vs_t)$, $v_t = \mathtt{cons}(v_t'',\ vs_t)$ and, since $\models \phi\alpha = 0$, we also derive that $v_t'' = \mathtt{inl}\ v_t'$,

  $$(v_i',\ v_c',\ v_t',\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\Box(\varphi\tau)]\!] \tag{B}$$

  and

  $$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[n-1]^{\varphi\alpha}\ \varphi\tau]\!]$$

  $$\models 0 < n$$

  From the induction hypothesis we get

  $$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[n-1]^{\varphi\alpha}\ \Box(\varphi\tau)]\!] \tag{C}$$

  We can combine eq. (B) and eq. (C) facts to derive that

  $$(\mathtt{cons}(v_i,\ vs_i),\ \mathtt{cons}(v_c,\ vs_c),\ \mathtt{cons}(\mathtt{inl}\ v_t,\ vs_t),\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[n]^{\varphi\alpha}\ \Box(\varphi\tau)]\!] \tag{A}$$

Case $\dfrac{\Delta; \Phi \models n \doteq n' \qquad \Delta; \Phi \models \alpha \leq \alpha' \leq n \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi \models^{\mathsf{A}} \mathtt{list}\,[n]^{\alpha}\ \tau \sqsubseteq \mathtt{list}\,[n']^{\alpha'}\ \tau'}$ **l1**

Let

$$(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[\varphi n]^{\varphi\alpha}\ \varphi\tau]\!]$$

Since $\models \varphi n = \varphi n'$ we derive

$$(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\mathtt{list}\,[\varphi n']^{\varphi\alpha}\ \varphi\tau]\!] \tag{A}$$

We will prove that for all α, β, n such that $\models \alpha \leq \beta \leq n$, if $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[\alpha]^n\ \varphi\tau]\!]$ then $(v_i,\ v_c,\ v_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[\beta]^n\ \varphi\tau']\!]$.

We proceed by subinduction on v_i.

- $v_i = \texttt{nil}$

  From eq. (A) we derive that $v_c = \texttt{nil}$, $v_t = \texttt{nil}$ and $\models n = 0$, $\models \alpha = 0$, and since $\models \alpha \leq \beta \leq n$, $\models \beta = 0$ From the definition of the relation we show that

  $$(\texttt{nil},\ \texttt{nil},\ \texttt{nil},\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n]^\beta\ \varphi\tau']\!]$$

- $v_i = \texttt{cons}(v_i',\ vs_i)$

  From eq. (A) we derive that $v_c = \texttt{cons}(v_t',\ vs_t)$, $v_t = \texttt{cons}(\texttt{inr}\ v_t',\ vs_t)$ or $v_t = \texttt{cons}(\texttt{inr}\ v_t',\ vs_t)$

  We consider the two cases separately.

  - $v_t = \texttt{cons}(\texttt{inl}\ v_t',\ vs_t)$

    From eq. (A) we derive

    $$(v_i',\ v_c',\ v_t',\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\Box(\varphi\tau)]\!]$$

    and

    $$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^\alpha\ \Box(\varphi\tau)]\!]$$

    $$\models n > 0$$

    or

    $$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[\varphi n - 1]^{\varphi\alpha - 1}\ \varphi\tau]\!]$$

    $$\models n > 0\ \wedge\ \models a > 0$$

    In the first case we can use the induction hypothesis on the premise of the rule to derive that

    $$(v_i',\ v_c',\ v_t',\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\Box(\varphi\tau')]\!]$$

    and the sub-induction hypothesis (note that (note that $\models \alpha - 1 \leq \beta - 1 \leq n - 1$) to derive that

    $$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^{\beta-1}\ \varphi\tau']\!]$$

    We can combine the two statements above in order to derive

    $$(\texttt{cons}(v_i,\ vs_i),\ \texttt{cons}(v_c,\ vs_c),\ \texttt{cons}(\texttt{inl}\ v_t,\ vs_t),\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n]^\beta\ \varphi\tau']\!]$$

    The other case is similar.

– $v_t = \texttt{cons}(\texttt{inr}\ v'_t,\ vs_t)$

From eq. (A) we derive

$$(v'_i,\ v'_c,\ v'_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\varphi\tau]\!]$$

and

$$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^{\alpha-1}\ \Box(\varphi\tau)]\!]$$
$$\models n > 0\ \wedge\ \models a > 0$$

We can use the induction hypothesis on the premise of the rule to derive that

$$(v'_i,\ v'_c,\ v'_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\varphi\tau']\!]$$

and the sub-induction hypothesis to derive that (note that $\models \alpha - 1 \leq \beta - 1 \leq n - 1$)

$$(vs_i,\ vs_c,\ vs_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n-1]^{\beta-1}\ \varphi\tau']\!]$$

We can combine the two statements above in order to derive

$$(\texttt{cons}(v_i,\ vs_i),\ \texttt{cons}(v_c,\ vs_c),\ \texttt{cons}(\texttt{inl}\ v_t,\ vs_t),\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{V}[\![\texttt{list}\,[n]^{\beta}\ \varphi\tau']\!]$$

□

□

Theorem 44 (Fundamental theorem - Unary interpretation)
Assume that the following hold:

$$\Delta; \Phi; \Gamma \vdash_{\mathbb{C}} e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner$$
$$\varphi \in \mathcal{D}[\![\Delta]\!]$$
$$(\theta_s,\ \theta_t,\ (\sigma,\ \beta,\ m)) \in \mathcal{G}(\!|\varphi\Gamma|\!)$$
$$\models \varphi\Phi$$
$$\texttt{wf}\ \sigma$$

Then

$$(\theta_s e,\ \theta_t \ulcorner e \urcorner,\ (\sigma,\ \beta,\ m)) \in \mathcal{E}(\!|\varphi\tau|\!)$$

Proof. We proceed by induction on the typing derivation of e. For each of the following cases we pick φ, θ_s, θ_t and $W = (\sigma_i,\ \beta,\ m)$ such that:

- $\varphi \in \mathcal{D}[\![\Delta]\!]$
- $\models \varphi\Phi$
- $(\theta_s,\ \theta_t,\ W) \in \mathcal{G}(\!|\varphi\Gamma|\!)$
- $\texttt{dom}(\beta) \subseteq \texttt{dom}(\sigma_i) \wedge \texttt{dom}'(\beta) \subseteq \texttt{dom}(\sigma_c)$

- $\texttt{wf}\ \sigma_i \wedge \texttt{wf}\ \sigma_c \wedge \texttt{FL}(\theta_t) \subseteq \sigma$

We show the most important cases bellow. The proof of introduction and elimination rules are largely similar, so we will only show few of them.

$$\frac{}{\Delta; \Phi; \Gamma, x : \tau \vdash_{\mathbb{C}} x : \tau \mid c_{var}() \hookrightarrow \texttt{read}(x,\ x.\ x)}\ \mathbf{var}_{\mathbb{C}}$$

We need to show

$$(\theta_s x,\ \texttt{read}(\theta_t x,\ x.\ x),\ W) \in \mathcal{E}(\!|(\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa)} \varphi\tau_2)^{\mathbb{S}}|\!)^{c_{var}()}$$

We pick arbitrary $v_i, \sigma_i, \beta, t_0, l$ such that $(\sigma_i,\ \beta,\ c_{var}()) > W$ and $\sigma_i(l) = \square$. From the definition of $\mathcal{G}(\!|\cdot|\!)$ we derive that $(\theta_s x,\ \theta_t x,\ (\sigma_i,\ \beta,\ m)) \in \mathcal{G}(\!|\varphi\tau|\!)$. Thus, $\theta_t x = l_t$ and $\sigma_i(\beta(l_t)) = (v,\ \vec{e})$ for some l_t, v and $\vec{e}$. We can show the goals. We distinguish two cases

- $r = \mathbb{S}$

  1. We can easily derive that
  $$\theta_s x \Downarrow \theta_s x$$
  2. From the definition of the evaluation relation we derive
  $$\theta_t x,\ \sigma_i,\ t_0 \Downarrow^{\mathbb{S}}_{L,\beta} l_n,\ \sigma_i[l_t \mapsto (v,\ (l,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l_n \mapsto (v,\ [])],\ t_0 + 2,\ 1$$
  3. It trivially follows that
  $$\models 1 \leq 1$$
  4. From the definition of the relation and Lemma 36 (note that $l_n \notin \texttt{dom}(\sigma_i)$ and $\sigma_i[l_t \mapsto (v,\ (l,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l_n \mapsto (v,\ [])] \sqsupseteq \sigma_i$) we can derive that
  $$(\theta_s x,\ l_n,\ (\sigma_i[l_t \mapsto (v,\ (l,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})],\ \beta,\ m - 1)) \in \mathcal{V}(\!|\varphi\tau|\!)$$
  5. Assume that $r = \mathbb{C}(l)$. From the definition of the relation and Lemma 36 (note that $\sigma_i[l_t \mapsto (v,\ (l,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l \mapsto (v,\ [])] \sqsupseteq \sigma_i$) we can derive that
  $$(\theta_s x,\ l,\ (\sigma_i[l_t \mapsto (v,\ (l,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l \mapsto (v,\ [])],\ \beta,\ m - 1)) \in \mathcal{V}(\!|\varphi\tau|\!)$$

- $r = \mathbb{C}(l)$

  1. We can easily derive that
  $$\theta_s x \Downarrow \theta_s x$$
  2. From the definition of the evaluation relation we derive
  $$\theta_t x,\ \sigma_i,\ t_0 \Downarrow^{\mathbb{C}(l)}_{L,\beta} l_n,\ \sigma_i[l_t \mapsto (v,\ (l,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l_n \mapsto (v,\ [])],\ t_0 + 2,\ 1$$
  3. It trivially follows that
  $$\models 1 \leq 1$$
  4. From the definition of the relation and Lemma 36 (note that $\sigma_i(l) = \square$ and $\sigma_i[l_t \mapsto (v,\ (l,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l \mapsto (v,\ [])] \sqsupseteq \sigma_i$) we can derive that
  $$(\theta_s x,\ l,\ (\sigma_i[l_t \mapsto (v,\ (l,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l \mapsto (v,\ [])],\ \beta,\ m - 1)) \in \mathcal{V}(\!|\varphi\tau|\!)$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}} \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta;\Phi;\Gamma \vdash_\epsilon e_2 : \tau_1 \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \qquad \epsilon \leq \delta \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + c_{app}(\epsilon, \mathbb{S})\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon e_1\ e_2 : \tau_2 \mid \kappa \hookrightarrow !\ulcorner e_1 \urcorner\ \ulcorner e_2 \urcorner}\ \mathbf{app}_{\mathbb{S}}$$

By the induction hypothesis applied on the premises we get:

$$(\theta_s e_1,\ \theta_t \ulcorner e_1 \urcorner,\ W) \in \mathcal{E}(\!|(\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2)^{\mathbb{S}}|\!)^{\varphi\kappa_1} \tag{IH1}$$

$$(\theta_s e_2,\ \theta_t e_2,\ W) \in \mathcal{E}(\!|\varphi\tau_1|\!)^{\varphi\kappa_2} \tag{IH2}$$

We pick arbitrary v_i, σ_i, β, t_0, r such that

$$(\sigma_i,\ \beta,\ \kappa_1 + \kappa_2 + \kappa_3 + c_{app}(\mathbb{C},\ \mathbb{S},)) > W$$

and

$$r = \mathbb{C}(l_n) \Rightarrow \sigma_i(l_n) = \square$$

We instantiate eq. (IH1) with σ_i, β, t_0 and $\mathbb{S}$ and we derive (note that $\kappa_1 < m$)

$$\theta_s e_1 \Downarrow v_{s1} \tag{A1}$$

$$\theta_t \ulcorner e_1 \urcorner,\ \sigma_i,\ t_0 \Downarrow^{\mathbb{S}}_{L,\beta} v_t,\ \sigma_1,\ t_1,\ c_1 \tag{A2}$$

$$\models c_1 \leq \kappa_1 \tag{A3}$$

$$(v_{i1},\ l_f,\ (\sigma_1,\ \beta,\ m - j_1)) \in \mathcal{V}(\!|(\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2)^{\mathbb{S}}|\!)$$

From the last statement we derive that $v_{i1} = \texttt{fix}\ f(x).\, e_i$, $\sigma_1(\beta(l_f)) = (\texttt{fix}\ f(x).\, e_t,\ \vec{e})$ and

$$(\texttt{fix}\ f(x).\, e_i,\ \texttt{fix}\ f(x).\, e_t,\ (\sigma_1,\ \beta,\ m - c_1)) \in \mathcal{V}_A(\!|\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2|\!) \tag{A4}$$

We instantiate eq. (IH2) with σ_1, β, t_0 and $\mathbb{S}$ (note that $\sigma_1 \sqsupseteq \sigma_i$ and $\kappa_2 < m$) and we derive

$$\theta_s e_1 \Downarrow v'_s \tag{B1}$$

$$\theta_t \ulcorner e_2 \urcorner,\ \sigma_1,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} v'_t,\ \sigma_2,\ t_2,\ c_2 \tag{B2}$$

$$\models c_2 \leq \kappa_2 \tag{B3}$$

$$(v'_s,\ v'_t,\ (\sigma_2,\ \beta,\ m - c_2)) \in \mathcal{V}(\!|\varphi\tau_2|\!)$$

Using Lemma 36 we can derive

$$(v'_s,\ v'_t,\ (\sigma_2,\ \beta,\ m - c_1 - c_2 - 1)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{B4}$$

We now instantiate eq. (A4) with $(\sigma_2,\ \beta,\ m - c_1 - c_2 - 2)$ and eq. (B4) and we obtain

$$(e'_s[x/v'_s][f/\ldots],\ e'_t[x/v'_t][f/\ldots],\ (\sigma_2,\ \beta,\ m - c_1 - c_2 - 2)) \in \mathcal{E}(\!|\varphi\tau_2|\!)^{\varphi\kappa'} \tag{C}$$

We instantiate eq. (C) with σ_3, β, t_2 and r (note that $\kappa' < m - c_1 - c_2 - 2$) to derive that

$$e'_s[x/v'_s][f/\ldots] \Downarrow v_s \tag{C1}$$

$$e'_t[x/v'_t][f/\ldots],\ \sigma_2,\ t_2 \Downarrow^{r}_{L,\beta} v_t,\ \sigma_3,\ t_3,\ c_3 \tag{C2}$$

$$\models c_3 \leq \kappa' \tag{C3}$$

$$r = \mathbb{S} \Rightarrow (v_s,\ v_t,\ (\sigma_3,\ \beta,\ m - c_1 - c_2 - c_3 - 2)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{C4}$$

$$r = \mathbb{C}(l_n) \Rightarrow (v_s,\ l_n,\ (\sigma_3[l_n \mapsto (v_t,\ [])],\ \beta,\ m - c_1 - c_2 - c_3 - 2)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{C5}$$

We can now show the goals.

1. From eq. (A1), eq. (B1) and eq. (C1) we derive

$$\theta_s(e_1\ e_2) \Downarrow v_s$$

2. From eq. (A2), eq. (B2) and eq. (C2) we derive

$$\theta_t(!e_1\ e_2),\ \sigma_i,\ t_0 \Downarrow^r_{L,\beta} v_t,\ \sigma_3,\ t_3,\ c_1 + c_2 + c_3 + 2$$

3. From eq. (A3), eq. (B3) and eq. (C3) we derive (note that $c_{app}(\mathbb{C}, \mathbb{S}) = 2$)

$$\models c_1 + c_2 + c_3 + 2 \leq \kappa_1 + \kappa_2 + \kappa' + c_{app}(\mathbb{C}, \mathbb{S})$$

4. From eq. (C4) it immediately follows that

$$r = \mathbb{S} \Rightarrow (v_s,\ v_t,\ (\sigma_3,\ \beta,\ m - c_1 - c_2 - c_3 - 2)) \in \mathcal{V}(\!|\varphi\tau_2|\!)$$

5. It is the case that $\sigma_3[l_n \mapsto (v_t,\ [])] \sqsupseteq \sigma_3$ From eq. (C5) and Lemma 36 it immediately follows that

$$r = \mathbb{C}(l_n) \Rightarrow (v_s,\ l_n,\ (\sigma_3[l_n \mapsto (v_t,\ [])],\ \beta,\ m - c_1 - c_2 - c_3 - 2)) \in \mathcal{V}(\!|\varphi\tau_2|\!)$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash_\epsilon e_1 : (\tau_1 \xrightarrow{\mathbb{C}(\kappa')} \tau_2)^{\mathbb{C}} \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta; \Phi; \Gamma \vdash_\epsilon e_2 : \tau_1 \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \qquad \models \mathbb{C} \trianglelefteq \tau_2 \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + c_{app}(\epsilon, \mathbb{C})\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon e_1\ e_2 : \tau_2 \mid \kappa \hookrightarrow \texttt{let}\, f\ =\ \ulcorner e_1 \urcorner \texttt{in let}\, x\ =\ \ulcorner e_2 \urcorner \texttt{in read}(f,\ f.\ f\ x)}\ \mathbf{app}_{\mathbb{C}}$$

By the induction hypothesis applied on the premises we get:

$$(\theta_s e_1,\ \theta_t \ulcorner e_1 \urcorner,\ W) \in \mathcal{E}(\!|(\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2)^{\mathbb{C}}|\!)^{\varphi\kappa_1} \tag{IH1}$$

$$(\theta_s e_2,\ \theta_t e_2,\ W) \in \mathcal{E}(\!|\varphi\tau_1|\!)^{\varphi\kappa_2} \tag{IH2}$$

We pick arbitrary v_i, σ_i, β, t_0, r such that

$$(\sigma_i,\ \beta,\ \kappa_1 + \kappa_2 + \kappa_3 + c_{app}(\mathbb{C},\ \mathbb{S},)) > W$$

and

$$r = \mathbb{C}(l_n) \Rightarrow \sigma_i(l_n) = \square$$

We instantiate eq. (IH1) with σ_i, β, t_0 and $\mathbb{S}$ and we derive (note that $\kappa_1 < m$)

$$\theta_s e_1 \Downarrow v_{s1} \tag{A1}$$

$$\theta_t \ulcorner e_1 \urcorner,\ \sigma_i,\ t_0 \Downarrow^{\mathbb{S}}_{L,\beta} v_t,\ \sigma_1,\ t_1,\ c_1 \tag{A2}$$

$$\models c_1 \leq \kappa_1 \tag{A3}$$

$$(v_{i1},\ l_f,\ (\sigma_1,\ \beta,\ m - j_1)) \in \mathcal{V}(\!|(\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2)^{\mathbb{S}}|\!)$$

From the last statement we derive that $v_{i1} = \texttt{fix}\ f(x).\, e_i$, $\sigma_1(\beta(l_f)) = (\texttt{fix}\ f(x).\, e_t,\ \vec{e})$ and

$$(\texttt{fix}\ f(x).\, e_i,\ \texttt{fix}\ f(x).\, e_t,\ (\sigma_1,\ \beta,\ m - c_1)) \in \mathcal{V}_A(\!|\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2|\!) \tag{A4}$$

We instantiate eq. (IH2) with σ_1, β, t_0 and $\mathbb{S}$ (note that $\sigma_1 \sqsupseteq \sigma_i$ and $\kappa_2 < m$) and we derive

$$\theta_s e_1 \Downarrow v'_s \tag{B1}$$

$$\theta_t{}^{\ulcorner}e_2{}^{\urcorner},\ \sigma_1,\ t_1 \Downarrow^{\mathbb{S}}_{L,\beta} v'_t,\ \sigma_2,\ t_2,\ c_2 \tag{B2}$$

$$\models c_2 \leq \kappa_2 \tag{B3}$$

$$(v'_s,\ v'_t,\ (\sigma_2,\ \beta,\ m - c_2)) \in \mathcal{V}(\!|\varphi\tau_2|\!)$$

Using Lemma 36 we can derive

$$(v'_s,\ v'_t,\ (\sigma_2,\ \beta,\ m - c_1 - c_2 - 4)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{B4}$$

We now instantiate eq. (A4) with $(\sigma_2,\ \beta,\ m - c_1 - c_2 - 4)$ and eq. (B4) and we obtain

$$(e'_s[x/v'_s][f/\ldots],\ e'_t[x/v'_t][f/\ldots],\ (\sigma_2,\ \beta,\ m - c_1 - c_2 - 4)) \in \mathcal{E}(\!|\varphi\tau_2|\!)^{\varphi\kappa'} \tag{C}$$

We instantiate eq. (C) with σ_3, β, t_2 and r (note that $\kappa' < m - c_1 - c_2 - 42$) to derive that

$$e'_s[x/v'_s][f/\ldots] \Downarrow v_s \tag{C1}$$

$$e'_t[x/v'_t][f/\ldots],\ \sigma_2,\ t_2 \Downarrow^{r}_{L,\beta} v_t,\ \sigma_3,\ t_3,\ c_3 \tag{C2}$$

$$\models c_3 \leq \kappa' \tag{C3}$$

$$r = \mathbb{S} \Rightarrow (v_s,\ v_t,\ (\sigma_3,\ \beta,\ m - c_1 - c_2 - c_3 - 4)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{C4}$$

$$r = \mathbb{C}(l_n) \Rightarrow (v_s,\ l_n,\ (\sigma_3[l_n \mapsto (v_t,\ [])],\ \beta,\ m - c_1 - c_2 - c_3 - 4)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{C5}$$

We can now show the goals. We consider two cases.

- $r = \mathbb{S}$

  1. From eq. (A1), eq. (B1) and eq. (C1) we derive

     $$\theta_s(e_1\ e_2) \Downarrow v_s$$

  2. From eq. (A2), eq. (B2) and eq. (C2) we derive

     $$\theta_t(\ulcorner e \urcorner),\ \sigma_i,\ t_0 \Downarrow^{\mathbb{S}}_{L,\beta} l_n,\ \sigma_3[l_f \mapsto \cdots][l_n \mapsto (v_t,\ [])],\ t_3,\ c_1 + c_2 + c_3 + 4$$

     where $l_t \notin \mathtt{dom}(\sigma_2)$
  3. From eq. (A3), eq. (B3) and eq. (C3) we derive (note that $c_{app}(\mathbb{C}, \mathbb{C}) = 4$)

     $$\models c_1 + c_2 + c_3 + 4 \leq \kappa_1 + \kappa_2 + \kappa' + c_{app}(\mathbb{C}, \mathbb{C})$$

  4. It is the case that $l_n \notin \sigma_3$ and $\sigma_3[l_f \mapsto \cdots][l_t \mapsto (v_t,\ [])] \sqsupseteq \sigma_3$ From eq. (C4) and Lemma 36 it immediately follows that

     $$(v_s,\ v_t,\ (\sigma_3[l_f \mapsto \cdots][l_n \mapsto (v_t,\ [])],\ \beta,\ m - c_1 - c_2 - c_3 - 4)) \in \mathcal{V}(\!|\varphi\tau_2|\!)$$

- $r = \mathbb{C}(l)$

  1. From eq. (A1), eq. (B1) and eq. (C1) we derive

     $$\theta_s(e_1\ e_2) \Downarrow v_s$$

2. From eq. (A2), eq. (B2) and eq. (C2) we derive

$$\theta_t\ulcorner e\urcorner,\ \sigma_i,\ t_0 \Downarrow^{\mathbb{C}(l)}_{L,\beta} v_t,\ \sigma_3[l_f \mapsto \cdots],\ t_3,\ c_1 + c_2 + c_3 + 4$$

3. From eq. (A3), eq. (B3) and eq. (C3) we derive

$$\models c_1 + c_2 + c_3 + 4 \leq \kappa_1 + \kappa_2 + \kappa' + c_{app}(\mathbb{C}, \mathbb{C})$$

4. It is the case that $\sigma_3[l_f \mapsto \cdots][l \mapsto (v_t,\ [])] \sqsupseteq \sigma_3$ (note that $\sigma_3(l) = \square$). From eq. (C5) and Lemma 36 it immediately follows that

$$(v_s,\ l,\ (\sigma_3[l \mapsto (v_t,\ [])],\ \beta,\ m - c_1 - c_2 - c_3 - 4)) \in \mathcal{V}(\!|\varphi\tau_2|\!)$$

□

Theorem 45 (Fundamental theorem - Binary interpretation)
Assume that the following hold

$$\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner$$

$$\varphi \in \mathcal{D}[\![\Delta]\!]$$
$$(\theta_i,\ \theta_c,\ \theta_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{G}[\![\varphi\Gamma]\!]$$

.

$$\models \varphi\Phi$$
$$\mathtt{dom}(\beta) \subseteq \mathtt{dom}(\sigma_i) \wedge \mathtt{dom}'(\beta) \subseteq \mathtt{dom}(\sigma_c)$$
$$\mathtt{wf}\ \sigma_i \wedge \mathtt{wf}\ \sigma_c \wedge \mathtt{FL}(\theta_t) \subseteq \sigma_i$$

Then

$$(\theta_i e,\ \theta_c e,\ \theta_t\ulcorner e\urcorner,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{E}[\![\varphi\tau]\!]^{\varphi\kappa}$$

Proof. We proceed by induction on the typing derivation of e. For each of the following cases we pick φ, θ_i, θ_c, θ_t and $W = (\sigma_i,\ \sigma_c,\ \beta,\ m)$ such that:

- $\varphi \in \mathcal{D}[\![\Delta]\!]$
- $\models \varphi\Phi$
- $(\theta_i,\ \theta_c,\ \theta_t,\ W) \in \mathcal{G}[\![\varphi\Gamma]\!]$
- $\mathtt{dom}(\beta) \subseteq \mathtt{dom}(\sigma_i) \wedge \mathtt{dom}'(\beta) \subseteq \mathtt{dom}(\sigma_c)$
- $\mathtt{wf}\ \sigma_i \wedge \mathtt{wf}\ \sigma_c \wedge \mathtt{FL}(\theta_t) \subseteq \sigma$

We show the most important cases bellow. The proofs of introduction and elimination rules are largely similar, so we will only show few of them.

Case $\dfrac{}{\Delta; \Phi; \Gamma, x : \tau \vdash_{\mathbb{S}} x : \tau \mid 0 \hookrightarrow x}\ \mathbf{var}_{\mathbb{S}}$

We need to show

$$(\theta_i x,\ \theta_c x,\ \theta_t x,\ W) \in \mathcal{E}[\![\tau]\!]^0$$

From the hypotheses and the definition of $\mathcal{G}[\![\cdot]\!]$ we know that

$$(\theta_i x,\ \theta_c x,\ \theta_t x,\ W) \in \mathcal{V}[\![\tau]\!]$$

The result follows from Lemma 40.

Case $\dfrac{}{\Delta; \Phi; \Gamma, x : \tau \vdash_{\mathbb{C}} x : \tau \mid c_{var}() \hookrightarrow \mathtt{read}(x,\ x.\ x)}\ \mathbf{var}_{\mathbb{C}}$

We need to show

$$(\theta_i x,\ \theta_c x,\ \mathtt{read}(\theta_t x,\ x.\ x),\ W) \in \mathcal{E}[\![\varphi\tau]\!]^{c_{var}()}$$

We pick arbitrary v_i, j_i, σ_i, σ_c, β, σ_o, β_o, c', t_0 and r such that

$$\mathcal{D}(\mathtt{edges}(\sigma_i),\ \mathtt{dom}(\beta_o)),\ \sigma_i,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c' \tag{1}$$

$$(\sigma_i,\ \sigma_c,\ \beta,\ j_i) \geq W$$

$$\theta_i(e_1\ e_2) \Downarrow v_i$$

and

$$r = \mathbb{C}(l) \Rightarrow \sigma_i(l) = \square\ \wedge\ l \notin \mathtt{dom}(\beta)$$

By inversion of the evaluation relation we derive that $v_i = \theta_i x$ and that $j_i = 0$. From the hypotheses and the definition of $\mathcal{G}[\![\cdot]\!]$, we can derive using Lemma 36 that

$$(\theta_i x,\ \theta_c x,\ \theta_t x,\ (\sigma_i,\ \sigma_c,\ \beta,\ k)) \in \mathcal{V}[\![\tau]\!]$$

From this we derive that $\theta_t x = l$ for an $l \in \mathtt{dom}(\sigma_i)$. Let $\sigma_i(l) = (v,\ \vec{e})$. We consider the following cases.

- $l \in \mathtt{dom}(\beta)$
  In this case we derive that $\tau = (A)^{\mathbb{C}}$, $\sigma_c(\beta(l)) = (v',\ \vec{e'})$ and that

  $$(\theta_i x,\ v,\ (\sigma_i,\ \emptyset,\ k)) \in \mathcal{V}_A[\![A]\!] \tag{A}$$

  $$\forall k,\ (\theta_c x,\ v',\ (\sigma_i \uplus \sigma_c,\ \beta,\ k)) \in \mathcal{V}_A[\![A]\!] \tag{B}$$

  We consider the following cases.

  - $r = \mathbb{S}$
    We can show the goals
    1. $\theta_c x \Downarrow \theta_c x$

2. We can easily derive that

$$\texttt{read}(\theta_t x,\ x.\ x),\ t_0,\ \sigma_i \Downarrow^{\mathbb{S}}_{L_1,\emptyset} l_n,\ \sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\ x,\ t_0,\ t_0{+}1) :: \vec{e})][l_n \mapsto (v,\ [])],\ t_0{+}2,\ \_$$

Furthermore, we will show that $\texttt{inv}(\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\ x,\ t_0,\ t_0+1) :: \vec{e})][l_n \mapsto (v,\ [])],\ \beta_o,\ l_n)$

Proof. Let $l \in \hat{\mathcal{R}}_{(\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ \texttt{dom}(\beta_o))}(\texttt{FL}(l_n))$. Since $l \in \texttt{dom}(\beta)$, we can also show that $\texttt{path}(\texttt{dom}(\beta_o),\ \texttt{edges}(\sigma[l \mapsto \ldots][l_n \mapsto (v,\ [])]),\ l_n)$. We can derive by the definition that $l = l_n$. Also, using Lemma 29, we derive that there is no $(l'_s,\ l'_d,\ f',\ t'_1,\ t'_2) \in \texttt{edges}(\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])])$ such that $\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ l'_s)$ and $t'_1 < t_1, t_2 < t'_2$. We conclude that $l_n \in \texttt{trg}(\mathcal{D}(\texttt{edges}(\sigma_i[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \texttt{dom}(\beta_o)))$ □

3. First we will show that $\mathcal{D}(\texttt{edges}(\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\ x,\ t_0,\ t_0+1) :: \vec{e})][l_n \mapsto (v,\ [])]),\ \texttt{dom}(\beta_o)) = \mathcal{D}(\texttt{edges}(\sigma_i),\ \texttt{dom}(\beta_o)) +\!\!+ [(l,\ l_n,\ \lambda x.\ x,\ t_0,\ t_0+1)]$.

Proof. Note that

$$\mathcal{D}(\texttt{edges}(\sigma_i[l_\iota \to \ldots][l_n \mapsto (v,\ [])]),\ \beta_o) =$$

$$\mathcal{D}(\texttt{edges}(\sigma_i) \cup \{(l,\ l_n,\ \lambda f.\ f\ v'_t,\ t_2,\ t_3)\},\ \beta_o)$$

We know that $l \in \texttt{dom}(\beta_1)$ and consequently from Lemma 30 we can derive that $l \in \mathcal{D}(\texttt{edges}(\sigma_i),\ \texttt{dom}(\beta_o))$ and $\texttt{path}(\texttt{dom}(\beta_o),\ \texttt{edges}(\sigma_i),\ l)$. We can use Lemma 27 (all the other preconditions follow from the hypotheses) to derive the result. □

Using Lemma 33 and eq. (1) we derive

$$\mathcal{D}(\texttt{edges}(\sigma_i[l_\iota \to \ldots][l_n \mapsto (v,\ [])]),\ \beta_o),\ \sigma_i[l \mapsto \cdots :: \vec{e})][l_n \mapsto (v,\ [])],\ \sigma_o,\ \beta_o$$
$$\rightsquigarrow$$
$$\sigma_c[l'_n \mapsto (v',\ [])],\ \beta[l_n \mapsto l'_n],\ c' + 1$$

4. We can easily derive that

$$1 \leq c_{var}()$$

5. Using the fact that $\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\ x,\ t_0,\ t_0{+}1) :: \vec{e})] \sqsupseteq \sigma_i$ and that $l_n \notin \texttt{dom}(\sigma_i)$ we derive $\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\ x,\ t_0,\ t_0+1) :: \vec{e})][l_n \mapsto v] \sqsupseteq \sigma_i$. From the definition of the relation and eq. (A), eq. (B) and Lemma 36, we can show that

$$(\theta_i x,\ \theta_c x,\ l_n,\ (\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ \sigma_c[l'_n \mapsto (v',\ [])],\ \beta[l_n \mapsto l'_n],\ k))$$
$$\in \mathcal{V}[\![\tau]\!]$$

– $r = \mathbb{C}(l_n)$

We can show the goals

1. $\theta_c x \Downarrow \theta_c x$
2. We can derive that

$$\texttt{read}(\theta_t x,\ x.\ x),\ t_0,\ \sigma_i \Downarrow^{\mathbb{C}(l_n)}_{L_1,\ \emptyset} v,\ \sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})],\ t_0+2,\ \_$$

Furthermore, we will show that $\texttt{inv}(\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})][l_n \mapsto (v,\ [])],\ \beta_o,\ l_n)$

Proof. Let $l \in \hat{\mathcal{R}}_{(\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ \texttt{dom}(\beta_o))}(\texttt{FL}(l_n))$. Since $l \in \texttt{dom}(\beta)$, we can also show that $\texttt{path}(\texttt{dom}(\beta_o),\ \texttt{edges}(\sigma[l \mapsto \ldots][l_n \mapsto (v,\ [])]),\ l_n)$. We can derive by the definition that $l = l_n$. Also, using Lemma 29, we derive that there is no $(l'_s,\ l'_d,\ f',\ t'_1,\ t'_2) \in \texttt{edges}(\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])])$ such that $\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ l'_s)$ and $t'_1 < t_1, t_2 < t'_2$. We conclude that $l_n \in \texttt{trg}(\mathcal{D}(\texttt{edges}(\sigma_i[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \texttt{dom}(\beta_o)))$ □

3. We can show that $\mathcal{D}(\texttt{edges}(\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})]),\ \texttt{dom}(\beta_o)) = \mathcal{D}(\texttt{edges}(\sigma_i),\ \texttt{dom}(\beta_o)) +\!\!+ [(l,\ l_n,\ \lambda x.\, x,\ t_0,\ t_0+1)]$.

Proof. Note that

$$\mathcal{D}(\texttt{edges}(\sigma_i[l_i \to \ldots][l_n \mapsto (v,\ [])]),\ \beta_o) =$$

$$\mathcal{D}(\texttt{edges}(\sigma) \cup \{(l,\ l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3)\},\ \beta_o)$$

We know that $l \in \texttt{dom}(\beta)$ and consequently from Lemma 30 we can derive that $l \in \mathcal{D}(\texttt{edges}(\sigma_i),\ \texttt{dom}(\beta_o))$ and $\texttt{path}(\texttt{dom}(\beta_o),\ \texttt{edges}(\sigma_i),\ l)$. We can use Lemma 27 (all the other preconditions follow from the hypotheses) to derive the result. □

Then, using eq. (1), Lemma 33 and Lemma 32

$$\mathcal{D}(\texttt{edges}(\sigma_i[l_i \to \ldots][l_n \mapsto (v,\ [])]),\ \beta_o),\ \sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})],\ \sigma_o,\ \beta_o$$
$$\rightsquigarrow$$
$$\sigma_c[l'_n \mapsto (v',\ [])],\ \beta[l_n \mapsto l'_n],\ c'+1$$

4. We can easily derive that

$$1 \leq c_{var}()$$

5. Using the fact that $\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})] \sqsupseteq \sigma_i$ and that $\sigma_i(l_n) = \square$ we derive $\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})][l_n \mapsto v] \sqsupseteq \sigma_i$. From the definition

of the relation and eq. (A), eq. (B) and Lemma 36, we can show that

$$(\theta_i x,\ \theta_c x,\ l_n,\ (\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ \sigma_c[l'_n \mapsto (v',\ [])],\ \beta[l_n \mapsto l'_n],\ k)) \in \mathcal{V}[\![\tau]\!]$$

- $l \notin \mathtt{dom}(\beta)$

  Let $\tau = (A)^{\mu}$. We derive that

$$(\theta_i x,\ \theta_c x,\ v,\ (\sigma_i,\ \sigma_c,\ \beta,\ k)) \in \mathcal{V}_A[\![A]\!] \tag{A}$$

  We consider the following cases.

  - $r = \mathbb{S}$

    We can show the goals

    1. $\theta_c x \Downarrow \theta_c x$
    2. We can easily derive that

$$\mathtt{read}(\theta_t x,\ x.\, x),\ t_0,\ \sigma_i \Downarrow^{\mathbb{S}}_{L_1,\emptyset} l_n,\ \sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0{+}1) :: \vec{e})][l_n \mapsto (v,\ [])],\ t_0{+}2,\ \_$$

       Furthermore, we will show that $\mathtt{inv}(\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l_n \mapsto (v,\ [])],\ l_n,\ )$

       Proof. Let $l \in \hat{\mathcal{R}}_{(\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ \mathtt{dom}(\beta_o))}(\mathtt{FL}(l_n))$. Since $l \notin \mathtt{dom}(\beta)$ we can derive that $\neg\mathtt{path}(\mathtt{dom}(\beta_o),\ \mathtt{edges}(\sigma[l \mapsto \ldots][l_n \mapsto (v,\ [])]),\ l_n)$. By the definition, we obtain that $\hat{\mathcal{R}}_{(\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ \mathtt{dom}(\beta_o))}(\mathtt{FL}(l_n)) = \hat{\mathcal{R}}_{(\sigma_i,\ \mathtt{dom}(\beta_o))}(\theta_t x)$ and the result follows from the hypothesis that $\mathtt{inv}(\sigma_i,\ \beta_o,\ \theta_t x)$ □

    3. We can show that $\mathcal{D}(\mathtt{edges}(\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0{+}1) :: \vec{e})][l_n \mapsto v]),\ \mathtt{dom}(\beta_o)) = \mathcal{D}(\mathtt{edges}(\sigma_i),\ \mathtt{dom}(\beta_o))$.

       Proof. □

       Then from eq. (1) and Lemma 32 we derive

$$\mathcal{D}(\mathtt{edges}(\sigma_i[l \mapsto \ldots][l_n \mapsto v]),\ \mathtt{dom}(\beta_o)),\ \sigma_[l \mapsto \ldots][l_n \mapsto v],\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c'$$

    4. We can easily derive that

$$c' - c' \leq c_{var}()$$

    5. Using the fact that $\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0{+}1) :: \vec{e})] \sqsupseteq \sigma_i$ and that $l_n \notin \mathtt{dom}(\sigma_i)$ we derive $\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0 + 1) :: \vec{e})][l_n \mapsto v] \sqsupseteq \sigma_i$. From the definition

of the relation and eq. (A), Lemma 36 we can show that

$$(\theta_i x,\ \theta_c x,\ l_n,\ (\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0{+}1) :: \vec{e})][l_n \mapsto (v,\ [])],\ \sigma_c,\ \beta,\ k)) \in \mathcal{V}[\![\tau]\!]$$

– $r = \mathbb{C}(l_n)$

We can show the goals

1. $\theta_c x \Downarrow \theta_c x$
2. We can easily derive that

$$\texttt{read}(\theta_t x,\ x.\ x),\ t_0,\ \sigma_i \Downarrow^{\mathbb{C}(l_n)}_{L_1,\ \emptyset} v,\ \sigma[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})],\ t_0+2,\ \_$$

Proof. Let $l \in \hat{\mathcal{R}}_{(\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ \texttt{dom}(\beta_o))}(\texttt{FL}(l_n))$. Since $l \notin \texttt{dom}(\beta)$, we can derive that $\neg\texttt{path}(\texttt{dom}(\beta_o),\ \texttt{edges}(\sigma[l \mapsto \ldots][l_n \mapsto (v,\ [])]),\ l_n)$. By the definition, we obtain that $\hat{\mathcal{R}}_{(\sigma_i[l \mapsto \ldots][l_n \mapsto (v,\ [])],\ \texttt{dom}(\beta_o))}(\texttt{FL}(l_n)) = \hat{\mathcal{R}}_{(\sigma_i,\ \texttt{dom}(\beta_o))}(\theta_t x)$ and the result follows from the hypothesis that $\texttt{inv}(\sigma_i,\ \beta_o,\ \theta_t x)$ □

3. We can show that $\mathcal{D}(\texttt{edges}(\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0{+}1) :: \vec{e})][l_n \mapsto v]),\ \texttt{dom}(\beta_o)) = \mathcal{D}(\texttt{edges}(\sigma_i),\ \texttt{dom}(\beta_o))$.

Proof. □

Then from eq. (1) and Lemma 32 we derive

$$\mathcal{D}(\texttt{edges}(\sigma_i[l \mapsto \ldots][l_n \mapsto v]),\ \texttt{dom}(\beta_o)),\ \sigma_i[l \mapsto \ldots][l_n \mapsto v],\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c'$$

4. We can easily derive that

$$c' - c' \leq c_{var}()$$

5. Using the fact that $\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})] \sqsupseteq \sigma_i$ and that $\sigma_i(l_n) = \square$ we derive $\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})][l_n \mapsto v] \sqsupseteq \sigma_i$. Also note that $l_n \notin \texttt{trg}(\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0+1) :: \vec{e})][l_n \mapsto (v,\ [])]$ From the definition of the relation, eq. (A) and Lemma 36 we can show that

$$(\theta_i x,\ \theta_c x,\ l_n,\ (\sigma_i[l \mapsto (v,\ (l_n,\ \lambda x.\, x,\ t_0,\ t_0{+}1) :: \vec{e})][l_n \mapsto (v,\ [])],\ \sigma_c,\ \beta,\ k)) \in \mathcal{V}[\![\tau]\!]$$

Case $\dfrac{\Delta; \Phi \vdash \Gamma\ \texttt{wf} \qquad \kappa = (\epsilon \doteq \mathbb{C}\ ?\ c_{real}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon r : (\texttt{real})^{\mathbb{S}} \mid \kappa \hookrightarrow \texttt{ref}\ r}$ **real**

We need to show

$$(r,\ r,\ \texttt{ref}\ r,\ W) \in \mathcal{E}[\![(\texttt{real})^{\mathbb{S}}]\!]^0$$

From Lemma 40 it suffices to show that

$$(r,\ r,\ r,\ W) \in \mathcal{V}[\![\texttt{real}]\!]$$

which follows from the definition of the relation.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_\epsilon e : (\tau_1 \times \tau_2)^{\mathbb{S}} \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = \kappa' + c_{fst}(\epsilon, \mathbb{S})}{\Delta; \Phi; \Gamma \vdash_\epsilon \texttt{fst}\ e : \tau_1 \mid \kappa \hookrightarrow \texttt{fst}\ !\ulcorner e \urcorner}\ \textbf{fst}_{\mathbb{S}}$

By the induction hypothesis applied on the premises we get:

$$(\theta_i e,\ \theta_c e,\ \theta_t \ulcorner e \urcorner,\ W) \in \mathcal{E}[\![(\varphi\tau_1 \times \varphi\tau_2)^{\mathbb{S}}]\!]^{\varphi\kappa'} \tag{IH}$$

We pick arbitrary v_i, j_i, σ_i, σ_c, β, σ_o, β_o, c', t_0 and r such that

$$\mathcal{D}(\texttt{edges}(\sigma_i),\ \texttt{dom}(\beta_o)),\ \sigma_i,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c' \tag{1}$$

$$(\sigma_i,\ \sigma_c,\ \beta,\ j_i) \geq W$$

$$\theta_i(\texttt{fst}\ e) \Downarrow v_i$$

and

$$r = \mathbb{C}(l) \Rightarrow \sigma_i(l) = \square\ \wedge\ l \notin \texttt{dom}(\beta)$$

By inversion of the evaluation relation we derive the following:

$$\theta_i e \Downarrow (v_{i1}, v_{i2}), j_1 \tag{2}$$

and $v_i = v_{i1}$, $j_i = j_1 + 1$ for some v_i and j_1. We instantiate eq. (IH) with (v_{i1}, v_{i2}), j_1, σ_i, σ_c, β, (note that $(\sigma_i,\ \sigma_c,\ \beta,\ j_1) \geq W$), σ_o, β_o, c', $\mathbb{S}$, t_0, eq. (1) and eq. (2) and we obtain v_c, l, $(\sigma_1,\ \sigma_1',\ \beta_1) \geq (\sigma_i,\ \sigma_c,\ \beta)$, t_1, c_1 such that:

$$\theta_c e_1 \Downarrow v_c \tag{A1}$$

$$\theta_t \ulcorner e_1 \urcorner,\ \sigma_i,\ t_0 \Downarrow^r_{L_1,\emptyset} l,\ \sigma_1,\ t_1,\ \_ \tag{A2}$$

$$(r = \mathbb{S} \Rightarrow \texttt{inv}(\sigma_1,\ \beta_o,\ v_t))\ \wedge\ (r = \mathbb{C}(l) \Rightarrow \texttt{inv}(\sigma_1[l \mapsto v_t],\ \beta_o,\ l)) \tag{A3}$$

$$\mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o)),\ \sigma_1,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_1',\ \beta_1,\ c_1 \tag{A4}$$

$$c_1 - c' \leq \varphi\kappa_1 \tag{A5}$$

$$r = \mathbb{S} \Rightarrow (v_i,\ v_c,\ l,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_1)) \in \mathcal{V}[\![(\varphi\tau_1 \times \varphi\tau_2)^{\mathbb{S}}]\!]$$

$$r = \mathbb{S} \Rightarrow (v_i,\ v_c,\ l,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_1)) \in \mathcal{V}[\![(\varphi\tau_1 \times \varphi\tau_2)^{\mathbb{S}}]\!]$$

$$r = \mathbb{C}(l) \Rightarrow (v_{i1},\ v_{c1},\ l,\ (\sigma_1[l \mapsto v_t],\ \sigma_1',\ \beta_1,\ m - j_1)) \in \mathcal{V}[\![(\varphi\tau_1 \times \varphi\tau_2)^{\mathbb{S}}]\!]$$

From the last statements we derive that $v_c = (v_{c1}, v_{c2})$, $\sigma_1(l) = (v_{t1}, v_{t2})$, $l \notin \texttt{dom}(\beta_1)$ and

$$(v_{i1},\ v_{c1},\ v_{t1},\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_1)) \in \mathcal{V}[\![\varphi\tau_1]\!] \tag{A6}$$

$$(v_{i1},\ v_{c1},\ v_{t1},\ (\sigma_1[l \mapsto (v_t,\ [])],\ \sigma_1',\ \beta_1,\ m - j_1)) \in \mathcal{V}[\![\varphi\tau_1]\!] \tag{A7}$$

We can now show the goals.

1. From eq. (A1) we can derive that

$$\theta_c(\texttt{fst}\ e) \Downarrow v_{c1}$$

2. From eq. (A2) we can derive that

$$\theta_t\ulcorner e\urcorner,\ \sigma_i,\ t_0 \Downarrow^r_{L_1,\emptyset} v_{t1},\ \sigma_1,\ t_1,\ \_$$

   Assume $r = \mathbb{S}$. Since $l \notin \texttt{dom}(\beta_1)$ we can show that $\neg\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_1,\ l)$ and $\hat{\mathcal{R}}_{(\sigma_1,\ \beta_o)}(\texttt{FL}(l)) = \hat{\mathcal{R}}_{(\sigma,\ \beta)}(\texttt{FL}(v))$. From eq. (A3) we obtain $\texttt{inv}(\sigma_1,\ \beta_o,\ v_{t1}))$ Assume $r = \mathbb{C}(l)$. Since $l \notin \texttt{dom}(\beta_1)$ we can show that $\neg\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_1,\ l)$ and $\hat{\mathcal{R}}_{(\sigma_1[l\mapsto(v_t,\ [])],\ \beta_o)}(\texttt{FL}(l)) = \hat{\mathcal{R}}_{(\sigma_1[l\mapsto(v_t,\ [])],\ \beta)}(\texttt{FL}(v))$. From eq. (A3) we obtain $\texttt{inv}(\sigma_1[l \mapsto (v_t,\ [])],\ \beta_o,\ v_{t1}))$.

3. From eq. (A4) we derive

$$\mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_o),\ \sigma_3,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_3',\ \beta_3,\ c_3$$

4. From eq. (A5) we can derive that

$$c_1 - c' \leq \varphi\kappa'$$

5. Assume that $r = \mathbb{S}$. From eq. (A6) we derive that

$$(v_{i1},\ v_{c1},\ v_{t1},\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_1]\!]$$

6. Assume that $r = \mathbb{C}(l)$. From eq. (A7) we derive that

$$(v_{i1},\ v_{c1},\ l,\ (\sigma_1[l \mapsto (v_t,\ [])],\ \sigma_1',\ \beta_1,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_1]\!]$$

Case $\dfrac{\Delta; \Phi; \Gamma, f : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}}, x : \tau_1 \vdash_\delta e : \tau_2 \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \kappa = (\epsilon = \mathbb{C}\ \texttt{?}\ c_{fix}()\ :\ 0)}{\Delta; \Phi; \Gamma \vdash_\epsilon \texttt{fix}\ f(x).\, e : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}} \mid \kappa \hookrightarrow \texttt{ref}\ (\texttt{fix}\ f(x).\ulcorner e \urcorner)}$ **fix1**

We need to show

$$(\theta_i(\texttt{fix}\ f(x).\,e),\ \theta_c(\texttt{fix}\ f(x).\,e),\ \texttt{ref}\ \theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner),\ W) \in \mathcal{E}[\![(\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa)} \varphi\tau_2)^{\mathbb{S}}]\!]^0$$

From Lemma 40 it suffices to show that

$$(\theta_i(\texttt{fix}\ f(x).\,e),\ \theta_c(\texttt{fix}\ f(x).\,e),\ \theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner),\ W) \in \mathcal{V}_A[\![\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa)} \varphi\tau_2]\!]$$

We will first show that

$$\forall k \geq m,\ (\theta_i(\texttt{fix}\ f(x).\,e),\ \theta_c(\texttt{fix}\ f(x).\,e),\ \theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner),\ (\sigma_i,\ \sigma_c,\ \beta,\ k)) \in \mathcal{V}_A[\![\varphi\tau_1 \xrightarrow{\mathbb{S}(\varphi(\kappa))} \varphi\tau_2]\!]$$

We proceed by induction on k.

- $k = 0$
  This is vacuous from the definition of $\mathcal{V}[\![\cdot]\!]$
- $k = k' + 1$
  From the induction hypothesis we know

  $$(\theta_i(\texttt{fix}\ f(x).\,e),\ \theta_c(\texttt{fix}\ f(x).\,e),\ \theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner),\ (\sigma_i,\ \sigma_c,\ \beta,\ k')) \in \mathcal{V}_A[\![\varphi\tau_1 \xrightarrow{\mathbb{S}(\varphi(\kappa))} \varphi\tau_2]\!] \tag{IH}$$

  We pick arbitrary $W' > (\sigma_i,\ \sigma_c,\ \beta,\ k)$ and v_i, v_c, v_t l such that $(v_i,\ v_c,\ v_t,\ W') \in \mathcal{V}[\![\tau_1]\!]$ and $\sigma_i(l) = \texttt{fix}\ f(e).\,\theta_t\ulcorner e\urcorner$. We can easily derive using Lemma 36 and the definition of the relation that

  $$(\theta_i[x \mapsto v_i, f \mapsto \texttt{fix}\ f(x).\,\theta_i e],\ \theta_c[x \mapsto v_c, f \mapsto \texttt{fix}\ f(x).\,\theta_c e],\ \theta_t[x \mapsto v_t, f \mapsto l],\ W')$$
  $$\in \mathcal{G}[\![\varphi\Gamma, f : (\varphi\tau_1 \xrightarrow{\mathbb{S}(\varphi\kappa)} \varphi\tau_2)^{\mathbb{S}}, x : \tau_1]\!]$$

  We instantiate the outer induction hypothesis with the above and we derive

  $$(\theta_i[x \mapsto v_i, f \mapsto \ldots]e,\ \theta_c[x \mapsto v_c, f \mapsto \ldots]e,\ \theta_t[x \mapsto v_t, f \mapsto \ldots]\ulcorner e\urcorner,\ W')$$
  $$\in \mathcal{E}[\![\varphi\tau_2]\!]^{\varphi\kappa}$$

  which proves the goal.

If $\delta = \mathbb{C}$ we also have to show that

$$(\theta_i(\texttt{fix}\ f(x).\,e),\ \theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner),\ (\sigma_i,\ \emptyset,\ m)) \in \mathcal{V}_A(\!|\varphi\tau_1 \xrightarrow{\mathbb{C}(\varphi(\kappa))} \varphi\tau_2|\!) \tag{1}$$

$$\forall\ k,\ (\theta_c(\texttt{fix}\ f(x).\,e),\ \theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner),\ (\sigma_i \uplus \sigma_c,\ \beta,\ k)) \in \mathcal{V}_A(\!|\varphi\tau_1 \xrightarrow{\mathbb{C}(\varphi(\kappa))} \varphi\tau_2|\!) \tag{2}$$

To show eq. (1) we will show

$$\forall k \geq m,\ (\theta_i(\texttt{fix}\ f(x).\, e),\ \theta_t(\texttt{fix}\ f(x).\, \ulcorner e \urcorner),\ (\sigma_i,\ \emptyset,\ k)) \in \mathcal{V}_A(\!|\varphi\tau_1 \xrightarrow{\mathbb{C}(\varphi(\kappa))} \varphi\tau_2|\!)$$

We proceed by induction on k.

- $k = 0$

  This case is vacuously true by the definition of $\mathcal{V}(\!|\cdot|\!)$
- $k = k' + 1$

  From the induction hypothesis we know

$$(\theta_i(\texttt{fix}\ f(x).\, e),\ \theta_y(\texttt{fix}\ f(x).\, \ulcorner e \urcorner),\ (\sigma_i,\ \emptyset,\ k')) \in \mathcal{V}_A(\!|\varphi\tau_1 \xrightarrow{\mathbb{C}(\varphi(\kappa))} \varphi\tau_2|\!) \qquad \text{(IH)}$$

  We pick arbitrary $W' > (\sigma_i,\ \emptyset,\ k')$ and $v_i,\ v_t\ l$, such that $(v_i,\ v_t,\ W') \in \mathcal{V}(\!|\tau_1|\!)$ and $\sigma(l) = \texttt{fix}\ f(x).\, \theta_t \ulcorner e \urcorner$. We can easily derive using the definition of the relation Lemma 36 that

$$(\theta_i[x \mapsto v_i, f \mapsto \texttt{fix}\ f(x).\, \theta_i e],\ \theta_t[x \mapsto v_t, f \mapsto l],\ W')$$
$$\in \mathcal{G}(\!|\varphi\Gamma, f : (\varphi\tau_1 \xrightarrow{\mathbb{C}(\varphi\kappa)} \varphi\tau_2)^{\mathbb{S}}, x : \tau_1|\!)$$

  We instantiate Theorem 44 with the above and we derive

$$(\theta_i[x \mapsto v_i, f \mapsto \texttt{fix}\ f(x).\, \theta_i e]e,\ \theta_t[x \mapsto v_t, f \mapsto \texttt{fix}\ f(x).\, l] \ulcorner e \urcorner,\ W') \in \mathcal{E}(\!|\varphi\tau_2|\!)^{\varphi\kappa}$$

  which proves the goal.

To show eq. (2) we pick arbitrary k, and we show

$$\forall k' \geq k,\ (\theta_c(\texttt{fix}\ f(x).\, e),\ \theta_t(\texttt{fix}\ f(x).\, \ulcorner e \urcorner),\ (\sigma_i \uplus \sigma_c,\ \beta,\ k')) \in \mathcal{V}_A(\!|\varphi\tau_1 \xrightarrow{\mathbb{C}(\varphi(\kappa))} \varphi\tau_2|\!)$$

following the same procedure as above.

Case $\dfrac{\begin{array}{c}\Delta; \Phi; \Gamma, f : \square((\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}}, x : \tau_1) \vdash_\delta e : \tau_2 \mid \kappa' \hookrightarrow \ulcorner e \urcorner \\ \forall x \in \Gamma,\quad \Delta; \Phi \models\models \Gamma(x) \sqsubseteq \square(\Gamma(x)) \qquad \kappa = (\epsilon = \mathbb{C}\ ?\ c_{fix}()\ :\ 0)\end{array}}{\Delta; \Phi; \Gamma \vdash_\epsilon \texttt{fix}\ f(x).\, e : \square((\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}}) \mid \kappa \hookrightarrow \texttt{ref}\ (\texttt{fix}\ f(x).\, \ulcorner e \urcorner)}$ **fix2**

We need to show

$$(\theta_i(\texttt{fix}\ f(x).\, e),\ \theta_c(\texttt{fix}\ f(x).\, e),\ \texttt{ref}\ \theta_t(\texttt{fix}\ f(x).\, \ulcorner e \urcorner),\ W) \in \mathcal{E}[\![\square((\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa)} \varphi\tau_2)^{\mathbb{S}})]\!]^0$$

We will first show that if

$$(\theta_i(\texttt{fix}\ f(x).\, e),\ \theta_c(\texttt{fix}\ f(x).\, e),\ \theta_t(\texttt{fix}\ f(x).\, \ulcorner e \urcorner),\ W) \in \mathcal{V}_A[\![\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa)} \varphi\tau_2]\!]$$

then for all $(\sigma_i,\ \sigma_c,\ \beta,\ k) \geq W$, l such that $\sigma_i(l) = \texttt{fix}\ f(x).\ulcorner e \urcorner$

$$(\theta_i(\texttt{fix}\ f(x).\, e),\ \theta_c(\texttt{fix}\ f(x).\, e),\ l,\ (\sigma_i,\ \sigma_c,\ \beta,\ k)) \in \mathcal{V}_A[\![\Box((\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa)} \varphi\tau_2)^{\mathbb{S}})]\!]$$

Proof. Let $W' = (\sigma_i,\ \sigma_c,\ \beta,\ k)$. We know that $(\theta_i,\ \theta_c,\ \theta_t,\ W) \in \mathcal{G}[\![\phi\Gamma]\!]$ Using Theorem 43 and Lemma 36 we can derive that

$$(\theta_i,\ \theta_c,\ \theta_t,\ W') \in \mathcal{G}[\![\Box(\phi\Gamma)]\!] \tag{1}$$

From Lemma 39, it suffices to show that $\mathcal{R}_{\sigma_i}(\texttt{FL}(\theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner))) \cap \texttt{dom}(\beta) = \emptyset$. From Lemma 34 we derive that $\texttt{FL}(\texttt{fix}\ f(x).\ulcorner e\urcorner) = \emptyset$. Consequently, $\texttt{FL}(\theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner)) = \texttt{FL}(\theta_t)$. From eq. (1) and the definition of the relations we can derive that $\texttt{FL}(\theta_t) \cap \texttt{dom}(\beta) = \emptyset$ that proves the goal. □

Using the above fact we can prove that

$$(\theta_i(\texttt{fix}\ f(x).\, e),\ \theta_c(\texttt{fix}\ f(x).\, e),\ \theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner),\ W) \in \mathcal{V}_A[\![\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa)} \varphi\tau_2]\!]$$

The proof is similar to the **fix1** case. From this, the fact we proved above, and Lemma 40 we can derive

$$(\theta_i(\texttt{fix}\ f(x).\, e),\ \theta_c(\texttt{fix}\ f(x).\, e),\ \texttt{ref}\ \theta_t(\texttt{fix}\ f(x).\ulcorner e\urcorner),\ W) \in \mathcal{E}[\![\Box((\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa)} \varphi\tau_2)^{\mathbb{S}})]\!]^0 \tag{2}$$

Case $$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : (\tau_1 \xrightarrow{\delta(\kappa')} \tau_2)^{\mathbb{S}} \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta;\Phi;\Gamma \vdash_\epsilon e_2 : \tau_1 \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \qquad \epsilon \leq \delta \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + c_{app}(\epsilon, \mathbb{S})\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon e_1\ e_2 : \tau_2 \mid \kappa \hookrightarrow\ !\ulcorner e_1 \urcorner\ \ulcorner e_2 \urcorner}\ \textbf{app}_{\mathbb{S}}$$

By the induction hypothesis applied on the premises we get:

$$(\theta_i e_1,\ \theta_c e_1,\ \theta_t \ulcorner e_1 \urcorner,\ W) \in \mathcal{E}[\![(\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa')} \varphi\tau_2)^{\mathbb{S}}]\!]^{\varphi\kappa_1} \tag{IH1}$$

$$(\theta_i e_2,\ \theta_c e_2,\ \theta_t \ulcorner e_2 \urcorner,\ W) \in \mathcal{E}[\![\varphi\tau_1]\!]^{\varphi\kappa_2} \tag{IH2}$$

We pick arbitrary v_i, j_i, σ_i, σ_c, β, σ_o, β_o, c', t_0 and r such that

$$\mathcal{D}(\texttt{edges}(\sigma_i),\ \texttt{dom}(\beta_o)),\ \sigma_i,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c' \tag{1}$$

$$(\sigma_i,\ \sigma_c,\ \beta,\ j_i) \geq W$$

$$\theta_i(e_1\ e_2) \Downarrow v_i$$

and

$$r = \mathbb{C}(l) \Rightarrow \sigma_i(l) = \Box\ \wedge\ l \notin \texttt{dom}(\beta)$$

By inversion of the evaluation relation we derive the following:

$$\theta_i e_1 \Downarrow \texttt{fix}\ f(x).\, e_i, j_1 \tag{2}$$

$$\theta_i e_2 \Downarrow v_i', j_2 \tag{3}$$

$$e_i[x/v_i'][f/\texttt{fix}\ f(x).\, e_i] \Downarrow v_i, j_3 \tag{4}$$

and $j_i = j_1 + j_2 + j_3$ for some e_i, v_i', j_1, j_2 and j_3 We instantiate eq. (IH1) with $\texttt{fix}\ f(x).\, e_i$, j_1, σ_i, σ_c, β, (note that $(\sigma_i,\ \sigma_c,\ \beta,\ j_1) \geq W$), σ_o, β_o, c', $\mathbb{S}$, t_0, eq. (1) and eq. (2) and we obtain v_{c1}, l_f, $(\sigma_1,\ \sigma_1',\ \beta_1) \geq (\sigma_i,\ \sigma_c,\ \beta)$, t_1, c_1 such that:

$$\theta_c e_1 \Downarrow v_{c1} \tag{A1}$$

$$\theta_t{}^{\ulcorner} e_1{}^{\urcorner},\ \sigma_i,\ t_0 \Downarrow^{\mathbb{S}}_{L_1,\emptyset} l_f,\ \sigma_1,\ t_1,\ \_ \tag{A2}$$

$$\texttt{inv}(\sigma_1,\ \beta_o,\ l_f) \tag{A3}$$

$$\mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o)),\ \sigma_1,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_1',\ \beta_1,\ c_1 \tag{A4}$$

$$c_1 - c' \leq \varphi\kappa_1 \tag{A5}$$

and

$$(\texttt{fix}\ f(x).\, e_i,\ v_{c1},\ l_f,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_1)) \in \mathcal{V}[\![(\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa')} \varphi\tau_2)^{\mathbb{S}}]\!]$$

From the last statement we obtain that $l_f \notin \texttt{trg}(\texttt{edges}(\sigma_1))$, $\sigma(l_f) = \texttt{fix}\ f(x).\, e_t$, $v_c = \texttt{fix}\ f(x).\, e_c$ and

$$(\texttt{fix}\ f(x).\, e_i,\ \texttt{fix}\ f(x).\, e_c,\ \texttt{fix}\ f(x).\, e_t,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_1)) \in \mathcal{V}[\![(\varphi\tau_1 \xrightarrow{\delta(\sigma\kappa')} \varphi\tau_2)^{\mathbb{S}}]\!] \tag{A6}$$

We now instantiate eq. (IH2) with v_i', j_2, σ_1, σ_1', β_1, (note that $(\sigma_1,\ \sigma_1',\ \beta_1,\ j_2) \geq W$), σ_o, β_o, c_1 t_1, $\mathbb{S}$ eq. (A4) and eq. (3) and we obtain v_c', v_t', $(\sigma_2,\ \sigma_2',\ \beta_2) \geq (\sigma_1,\ \sigma_1,\ \beta_1)$, t_2, c_2 such that:

$$\theta_c e_2 \Downarrow v_c' \tag{B1}$$

$$\theta_t{}^{\ulcorner} e_2{}^{\urcorner},\ \sigma_1,\ t_1 \Downarrow^{\mathbb{S}}_{L_1,\emptyset} v_t',\ \sigma_2,\ t_2,\ \_ \tag{B2}$$

$$\texttt{inv}(\sigma_2,\ \beta_o,\ v_t') \tag{B3}$$

$$\mathcal{D}(\texttt{edges}(\sigma_2),\ \texttt{dom}(\beta_o)),\ \sigma_2,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_2',\ \beta_2,\ c_2 \tag{B4}$$

$$c_2 - c_1 \leq \varphi\kappa_2 \tag{B5}$$

and

$$(v_i',\ v_c',\ v_t',\ (\sigma_2,\ \sigma_2',\ \beta_2,\ m - j_2)) \in \mathcal{V}[\![\varphi\tau_1]\!]$$

We apply Lemma 36 at the last statement and we obtain

$$(v_i',\ v_c',\ v_t',\ (\sigma_2,\ \sigma_2',\ \beta_2,\ m - j_1 - j_2 - 1)) \in \mathcal{V}[\![\varphi\tau_1]\!] \tag{B6}$$

We instantiate eq. (A6) with σ_2, σ_2', β_2, $m-j_1-j_2-1$ (note that $(\sigma_2,\ \sigma_2',\ \beta_2,\ m-j_1-j_2-1) > (\sigma_1,\ \sigma_1',\ \beta_1,\ m-j_1)$) by Lemma 29 and Lemma 30), and eq. (B6) and we derive

$$(e_i'[x/v_i'][f/\ldots],\ e_c'[x/v_c'][f/\ldots],\ e_t'[x/v_t'][f/\ldots],\ (\sigma_2,\ \sigma_2',\ \beta_2,\ m-j_1-j_2-1)) \in \mathcal{E}[\![\varphi\tau_2]\!]^{\varphi\kappa'} \tag{C}$$

We now instantiate eq. (C) with v_i, j_3, σ_2, σ_2', β_2, σ_o, β_o, c_2, t_2, r (note that $r = \mathbb{C}(l) \Rightarrow \sigma_2(l) = \square$ from Lemma 29), eq. (B4) and eq. (4) and we obtain v_c, v, $(\sigma_3,\ \sigma_3',\ \beta_3) \geq (\sigma_2,\ \sigma_2',\ \beta_2)$, t_3, c_3, such that:

$$e_c'[x/v_c'][f/\ldots] \Downarrow v_c \tag{C1}$$

$$e_t'[x/v_t'][f/\ldots],\ \sigma_2,\ t_2 \Downarrow^r_{L_1,\emptyset} v_t,\ \sigma_3,\ t_3,\ \_ \tag{C2}$$

$$(r = \mathbb{S} \Rightarrow \texttt{inv}(\sigma_3,\ \beta_o,\ v_t)) \ \wedge\ (r = \mathbb{C}(l) \Rightarrow \texttt{inv}(\sigma_3[l \mapsto v_t],\ \beta_o,\ l)) \tag{C3}$$

$$\mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_o),\ \sigma_3,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_3',\ \beta_3,\ c_3 \tag{C4}$$

$$c_3 \leq \varphi\kappa' \tag{C5}$$

$$r = \mathbb{S} \Rightarrow (v_i,\ v_c,\ v_t,\ (\sigma_3,\ \sigma_3',\ \beta_3,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_1]\!] \tag{C6}$$

$$r = \mathbb{C}(l) \Rightarrow (v_i,\ v_c,\ l,\ (\sigma_3[l \mapsto (v_t,\ [])],\ \sigma_3',\ \beta_3,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_1]\!] \tag{C7}$$

We can now show the goals.

1. From eq. (A1), eq. (B1) and eq. (C1) we can derive that

   $$\theta_c(e_1\ e_2) \Downarrow v_c$$

2. From eq. (A2), eq. (B2) and eq. (C2) we can derive that

   $$\theta_t{}^{\ulcorner}e^{\urcorner},\ \sigma_i,\ t_0 \Downarrow^r_{L_1,\emptyset} v_t,\ \sigma_3,\ t_3,\ \_$$

   Furthermore, from eq. (C3) we obtain

   $$(r = \mathbb{S} \Rightarrow \texttt{inv}(\sigma_3,\ \beta_o,\ v_t)) \ \wedge\ (r = \mathbb{C}(l) \Rightarrow \texttt{inv}(\sigma_3[l \mapsto v_t],\ \beta_o,\ l))$$

3. From eq. (C4) we derive

   $$\mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_o),\ \sigma_3,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_3',\ \beta_3,\ c_3$$

4. From eq. (A5), eq. (B5), and eq. (C5) we can derive that

   $$c_3 - c' \leq \varphi(\kappa_1 + \kappa_2 + \kappa' + (\epsilon = \mathbb{C}\ ?\ 2\ :\ 0))$$

5. Assume that $r = \mathbb{S}$. From eq. (C6) we derive that

$$(v_i,\ v_c,\ v_t,\ (\sigma_3,\ \sigma_3',\ \beta_3,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_2]\!]$$

6. Assume that $r = \mathbb{C}(l)$. From eq. (C7) we derive that

$$(v_i,\ v_c,\ l,\ (\sigma_3[l \mapsto (v_t,\ [])],\ \sigma_3',\ \beta_3,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_2]\!]$$

Case $\dfrac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash_\epsilon e_1 : (\tau_1 \xrightarrow{\mathbb{C}(\kappa')} \tau_2)^{\mathbb{C}} \mid \kappa_1 \hookrightarrow \ulcorner e_1 \urcorner \\ \Delta;\Phi;\Gamma \vdash_\epsilon e_2 : \tau_1 \mid \kappa_2 \hookrightarrow \ulcorner e_2 \urcorner \qquad \models \mathbb{C} \trianglelefteq \tau_2 \qquad \kappa = \kappa' + \kappa_1 + \kappa_2 + c_{app}(\epsilon, \mathbb{C})\end{array}}{\Delta;\Phi;\Gamma \vdash_\epsilon e_1\ e_2 : \tau_2 \mid \kappa \hookrightarrow \texttt{let}\, f\ =\ \ulcorner e_1 \urcorner \texttt{in let}\, x\ =\ \ulcorner e_2 \urcorner \texttt{in read}(f,\ f.\ f\ x)}\ \mathbf{app}_{\mathbb{C}}$

By the induction hypothesis applied on the premises we get:

$$(\theta_i e_1,\ \theta_c e_1,\ \theta_t \ulcorner e_1 \urcorner,\ W) \in \mathcal{E}[\![(\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2)^{\mathbb{C}}]\!]^{\varphi\kappa_1} \tag{IH1}$$

$$(\theta_i e_2,\ \theta_c e_2,\ \theta_t \ulcorner e_2 \urcorner,\ W) \in \mathcal{E}[\![\varphi\tau_1]\!]^{\varphi\kappa_2} \tag{IH2}$$

We pick arbitrary v_i, j_i, σ_i, σ_c, β, σ_o, β_o, c', t_0 and r such that

$$\mathcal{D}(\texttt{edges}(\sigma_i),\ \texttt{dom}(\beta_o)),\ \sigma_i,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_c,\ \beta,\ c' \tag{1}$$

$$(\sigma_i,\ \sigma_c,\ \beta,\ j_i) \geq W$$

$$\theta_i(e_1\ e_2) \Downarrow v_i$$

and

$$r = \mathbb{C}(l) \Rightarrow \sigma_i(l) = \square\ \wedge\ l \notin \texttt{dom}(\beta)$$

By inversion of the evaluation relation we derive the following:

$$\theta_i e_1 \Downarrow \texttt{fix}\ f(x).\, e_i, j_1 \tag{2}$$

$$\theta_i e_2 \Downarrow v_i', j_2 \tag{3}$$

$$e_i[x/v_i'][f/\texttt{fix}\ f(x).\, e_i] \Downarrow v_i, j_3 \tag{4}$$

and $j_i = j_1 + j_2 + j_3$ for some e_i, v_i', j_1, j_2 and j_3 We instantiate eq. (IH1) with $\texttt{fix}\ f(x).\, e_i$, j_1, σ_i, σ_c, β, (note that $(\sigma_i,\ \sigma_c,\ \beta,\ j_1) \geq W$), σ_o, β_o, c', $\mathbb{S}$, t_0, eq. (1) and eq. (2) and we obtain v_{c1}, l_f, $(\sigma_1,\ \sigma_1',\ \beta_1) \geq (\sigma_i,\ \sigma_c,\ \beta)$, t_1, c_1 such that:

$$\theta_c e_1 \Downarrow v_{c1} \tag{A1}$$

$$\theta_t \ulcorner e_1 \urcorner,\ \sigma_i,\ t_0 \Downarrow^{\mathbb{S}}_{L_1,\emptyset} l_f,\ \sigma_1,\ t_1,\ \_ \tag{A2}$$

$$\texttt{inv}(\sigma_1,\ \beta_o,\ l_f) \tag{A3}$$

$$\mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o)),\ \sigma_1,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_1',\ \beta_1,\ c_1 \tag{A4}$$

$$c_1 - c' \leq \varphi\kappa_1 \tag{A5}$$

and

$$(\texttt{fix}\ f(x).\, e_i,\ v_{c1},\ l_f,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_1)) \in \mathcal{V}[\![(\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2)^{\mathbb{C}}]\!] \tag{A6}$$

We now instantiate eq. (IH2) with v_i', j_2, σ_1, σ_1', β_1, (note that $(\sigma_1,\ \sigma_1',\ \beta_1,\ j_2) \geq W$), σ_o, β_o, $c_1\ t_1$, $\mathbb{S}$ eq. (A4) and eq. (3) and we obtain v_c', v_t', $(\sigma_2,\ \sigma_2',\ \beta_2) \geq (\sigma_1,\ \sigma_1,\ \beta_1)$, t_2, c_2 such that:

$$\theta_c e_2 \Downarrow v_c' \tag{B1}$$

$$\theta_t{}^{\ulcorner} e_2{}^{\urcorner},\ \sigma_1,\ t_1 \Downarrow^{\mathbb{S}}_{L_1,\emptyset} v_t',\ \sigma_2,\ t_2,\ \_ \tag{B2}$$

$$\texttt{inv}(\sigma_2,\ \beta_o,\ v_t') \tag{B3}$$

$$\mathcal{D}(\texttt{edges}(\sigma_2),\ \texttt{dom}(\beta_o)),\ \sigma_2,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_2',\ \beta_2,\ c_2 \tag{B4}$$

$$c_2 - c_1 \leq \varphi\kappa_2 \tag{B5}$$

and

$$(v_i',\ v_c',\ v_t',\ (\sigma_2,\ \sigma_2',\ \beta_2,\ m - j_2)) \in \mathcal{V}[\![\varphi\tau_1]\!]$$

We apply Lemma 36 and Lemma 37 at the last statement and we obtain

$$(v_i',\ v_c',\ v_t',\ (\sigma_2,\ \sigma_2',\ \beta_2,\ m - j_1 - j_2 - 1)) \in \mathcal{V}[\![\varphi\tau_1]\!] \tag{B6.1}$$

$$\forall\ m,\ (v_i',\ v_t',\ (\sigma_2,\ \emptyset,\ m)) \in \mathcal{V}(\!|\varphi\tau_1|\!) \tag{B6.2}$$

$$\forall\ m,\ (v_c',\ v_t',\ (\sigma_2 \uplus \sigma_2',\ \beta_2,\ m)) \in \mathcal{V}(\!|\varphi\tau_1|\!) \tag{B6.3}$$

We distinguish two cases.

- $l_f \notin \texttt{dom}(\beta_1)$
  From eq. (A6) we derive that $\sigma_1(l_f) = (\texttt{fix}\ f(x).\, e_t,\ \vec{e})$ and $v_c' = \texttt{fix}\ f(x).\, e_c$, for some e_c, $\vec{e}$, and e_t, and

  $$(\texttt{fix}\ f(x).\, e_i,\ \texttt{fix}\ f(x).\, e_c,\ \texttt{fix}\ f(x).\, e_t,\ (\sigma_1,\ \sigma_1',\ \beta_1,\ m - c_1)) \in \mathcal{V}_A[\![\varphi\tau_1 \xrightarrow{\mathbb{S}(\sigma\kappa')} \varphi\tau_2]\!] \tag{A6.1}$$

  We instantiate eq. (A6.1) with σ_2, σ_2', β_2, $m - j_1 - j_2 - 1$ (note that $(\sigma_2,\ \sigma_2',\ \beta_2,\ m - j_1 -$

$j_2 - 1) > (\sigma_1,\ \sigma_1',\ \beta_1,\ m - j_1)$ by Lemma 29), and eq. (B6.1) and we derive

$$(e_i'[x/v_i'][f/\ldots],\ e_c'[x/v_c'][f/\ldots],\ e_t'[x/v_t'][f/\ldots],\ (\sigma_2,\ \sigma_2',\ \beta_2,\ m - j_1 - j_2 - 1)) \in \mathcal{E}[\![\varphi\tau_2]\!]^{\varphi\kappa'} \tag{C}$$

We consider two cases.

- $r = \mathbb{S}$

  Let $l_n = \texttt{fresh}_{L_1}(\sigma_2)$. We instantiate eq. (C) with v_i, j_3, $\sigma_2[l_n \mapsto \square]$, σ_2', β_2, (note that $\sigma_2[l_n \mapsto \square] \sqsupseteq \sigma_2$ and $j_3 < m - j_1 - j_2 - 1$ and $l_n \notin \texttt{dom}(\beta_2)$ since $\texttt{dom}(\beta_2) \subseteq \texttt{dom}(\sigma_2)$) σ_o, β_o, c_2, $t_2 + 1$, $\mathbb{C}(l_n)$ eq. (B4) and eq. (4) and we obtain v_c, v, $(\sigma_3,\ \sigma_3',\ \beta_3) \geq (\sigma_2[l_n \mapsto \square],\ \sigma_2',\ \beta_2)$, t_3, c_3, such that:

$$e_c[x/v_c'][f/\ldots] \Downarrow v_c \tag{C1}$$

$$e_t'[x/v_t'][f/\ldots],\ \sigma_2[l_n \mapsto \square],\ t_2 + 1 \Downarrow_{L,\beta}^{\mathbb{C}(l_n)} v,\ \sigma_3,\ t_3,\ \_$$

  and consequently

$$(\texttt{fix}\ f(x).\, e_t)\ v_t',\ \sigma_2[l_n \mapsto \square],\ t_2 + 1 \Downarrow_{L,\beta}^{\mathbb{C}(l_n)} v,\ \sigma_3,\ t_3,\ \_ \tag{C2}$$

$$\texttt{inv}(\sigma_3[l_n \mapsto (v,\ [])],\ \beta_o,\ l_n) \tag{C3}$$

$$\mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_o),\ \sigma_3,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma_3',\ \beta_3,\ c_3 \tag{C4}$$

$$c_3 - c_2 \leq \varphi\kappa' \tag{C5}$$

$$(v_i,\ v_c,\ l_n,\ (\sigma_3[l_n \mapsto (v,\ [])],\ \sigma_3',\ \beta_3,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_1]\!] \tag{C6}$$

  We can now show the goals.

  1. From eq. (A1), eq. (B1) and eq. (C1) we can derive that

$$\theta_c(e_1\ e_2) \Downarrow v_c$$

  2. From eq. (A2), eq. (B2) and eq. (C2) we can derive that

$$\theta_t\ulcorner e\urcorner,\ \sigma_i,\ t_0 \Downarrow_{L_1,\emptyset}^{\mathbb{S}} l_n,\ \sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3) :: \vec{e})][l_n \mapsto (v,\ [])],\ t_4{+}1,\ \_$$

     We will show that $\neg\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ l_f)$.

     Proof. Assume that $\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ l_f)$. Then from Lemma 25 we derive that $\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_1,\ l_f)$. It is the case that $\hat{\mathcal{R}}_{(\sigma_1,\ \texttt{dom}(\beta_o))}(l_f)$ and using $\texttt{inv}(\sigma_1,\ l_f,\ )$ we derive that $l_f \in \mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o))$. Using Lemma 30 we derive that $l_f \in \texttt{dom}(\beta_1)$ which is contradictory. □

     We will also show that $\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3) :: \vec{e})][l_n \mapsto (v,\ [])]),\ \beta_o) = \mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_o)$

Proof. Since $l_f \notin \texttt{dom}(\beta_1)$ we derive that $l_f \notin \texttt{dom}(\beta_o)$. If $l_f \notin \texttt{trg}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]) \cup \texttt{dom}(\beta_o)$ the result follows by the definition of dependency graph. Assume that $l_f \in \texttt{trg}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])])$. Using Lemma 29 we derive that $l_f \in \texttt{trg}(\sigma_1)$ Since $\neg\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ l_f)$ the result follows from Lemma 26. □

Now, we can show that $\texttt{inv}(\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3) :: \vec{e})][l_n \mapsto (v,\ [])],\ \beta_o,\ l_n)$

Proof. We can easily show that $\hat{\mathcal{R}}_{(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ \texttt{dom}(\beta_o))}(\texttt{FL}(l_n)) = \hat{\mathcal{R}}_{(\sigma_3,\ \texttt{dom}(\beta_o))}(\texttt{FL}(l_n))$. Since $\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \beta_o) = \mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_o)$ the result follows from eq. (C3) □

3. From the facts we proved above and using eq. (C4) we conclude that

$$\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \beta_o),\ \sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ \sigma_o,\ \beta_o \rightsquigarrow \sigma'_3,\ \beta_3,\ c_3$$

4. From eq. (A5), eq. (B5), and eq. (C5) we can derive that

$$c_3 - c' \leq \varphi(\kappa_1 + \kappa_2 + \kappa' + c_{app}(\epsilon, \mathbb{S}))$$

5. Using the fact that $\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3) :: \vec{e})] \sqsupseteq \sigma_3$ we derive $\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3) :: \vec{e})][l_n \mapsto (v,\ [])] \sqsupseteq \sigma_3[l_n \mapsto (v,\ [])]$. From eq. (C6) and Lemma 36 we derive that

$$(v_i,\ v_c,\ l_n,\ (\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ \sigma'_3,\ \beta_3,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_2]\!]$$

– $r = \mathbb{C}(l_n)$

We instantiate eq. (C) with v_i, j_3, σ_2, σ'_2, β_2, (note that $j_3 < m - j_1 - j_2 - 1$, $\sigma_2(l_n) = \square$ and $l_n \notin \texttt{dom}(\beta_2)$ using the initial hypothesis and Lemma 29, Lemma 30) σ_o, β_o, c_2, $t_2 + 1$, $\mathbb{C}(l_n)$ eq. (B4) and eq. (4) and we obtain v_c, v, $(\sigma_3,\ \sigma'_3,\ \beta_3) \geq (\sigma_2,\ \sigma'_2,\ \beta_2)$, t_3, c_3, such that:

$$e'_c[x/v'_c][f/\ldots] \Downarrow v_c \tag{C1}$$

$$e'_t[x/v'_t][f/\ldots],\ \sigma_2,\ t_2 + 1 \Downarrow^{\mathbb{C}(l_n)}_{L_1,\ \emptyset} v,\ \sigma_3,\ t_3,\ \_ \tag{C2}$$

$$\texttt{inv}(\sigma_3[l_n \mapsto (v,\ [])],\ \beta_o,\ l_n) \tag{C3}$$

$$\mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_0),\ \sigma_3,\ \sigma_o,\ \beta_o \rightsquigarrow \sigma'_3,\ \beta_3,\ c_3 \tag{C4}$$

$$c_3 - c_2 \leq \varphi\kappa' \tag{C5}$$

$$(v_i,\ v_c,\ l_n,\ (\sigma_3[l_n \mapsto (v,\ [])],\ \sigma_3',\ \beta_3,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_1]\!] \tag{C6}$$

We can now show the goals.

1. From eq. (A1), eq. (B1) and eq. (C1) we can derive that

$$\theta_c(e_1\ e_2) \Downarrow v_c$$

2. From eq. (A2), eq. (B2) and eq. (C2) we can derive that

$$\theta_t{}^{\ulcorner} e^{\urcorner},\ \sigma_i,\ t_0 \Downarrow_{L_1,\ \emptyset}^{\mathbb{C}(l_n)} v,\ \sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3),\ \vec{(e)})],\ t_3 + 1,\ \_$$

We will show that $\neg\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_3[l_f \mapsto \ldots],\ l_f)$.

Proof. Assume that $\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_3[l_f \mapsto \ldots],\ l_f)$. Then from Lemma 25 we derive that $\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_1,\ l_f)$. It is the case that $\hat{\mathcal{R}}_{(\sigma_1,\ \texttt{dom}(\beta_o))}(l_f)$ and using $\texttt{inv}(\sigma_1,\ l_f,\ )$ we derive that $l_f \in \mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o))$. Using Lemma 30 we derive that $l_f \in \texttt{dom}(\beta_1)$ which is contradictory. □

We will also show that $\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3) :: \vec{e})]),\ \beta_o) = \mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_o)$

Proof. Since $l_f \notin \texttt{dom}(\beta_1)$ we derive that $l_f \notin \texttt{dom}(\beta_o)$. If $l_f \notin \texttt{trg}(\sigma_3[l_f \mapsto \ldots]) \cup \texttt{dom}(\beta_o)$ the result follows by the definition of dependency graph. Assume that $l_f \in \texttt{trg}(\sigma_3[l_f \mapsto \ldots])$. Using Lemma 29 we derive that $l_f \in \texttt{trg}(\sigma_1)$ Since $\neg\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_3[l_f \mapsto \ldots],\ l_f)$ the result follows from Lemma 26. □

Now, we can show that $\texttt{inv}(\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3) :: \vec{e})]][l_n \mapsto (v,\ [])],\ \beta_o,\ l_n)$

Proof. We ca show that $\hat{\mathcal{R}}_{(\sigma_3[l_f \mapsto \ldots]][l_n \mapsto (v,\ [])],\ \texttt{dom}(\beta_o))}(\texttt{FL}(v)) = \hat{\mathcal{R}}_{(\sigma_3,\ \texttt{dom}(\beta_o))}(\texttt{FL}(v))$ and since $\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \beta_o) = \mathcal{D}(\texttt{edges}(\sigma_3),\ \beta_o)$ the result follows from eq. (C3) □

3. From the facts we proved above and eq. (C4) we conclude that

$$\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots]),\ \beta_o),\ \sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ \sigma_o,\ \beta_o$$
$$\rightsquigarrow$$
$$\sigma_3',\ \beta_3,\ c_3$$

4. Using the fact that $\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3),\ \vec{(e)})] \sqsupseteq \sigma_3$ we derive $\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3),\ \vec{(e)})][l_n \mapsto (v,\ [])] \sqsupseteq \sigma_3[l_n \mapsto (v,\ [])]$.

From eq. (C6) and Lemma 36 we derive that

$$(v_i,\ v_c,\ l_n,\ (\sigma_3[l_f \mapsto (\ldots)][l_n \mapsto (v,\ [])],\ \sigma_3',\ \beta_3,\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_2]\!]$$

- $l_f \in \texttt{dom}(\beta_1)$

  From eq. (A6) we derive that $\sigma_1(l) = (\texttt{fix}\ f(x).\, e_{t1},\ \vec{e})$, $\sigma_1'(\beta_1(l)) = (\texttt{fix}\ f(x).\, e_{t2},\ \_)$ and $v_c' = \texttt{fix}\ f(x).\, e_c$, for some e_c, e_{t1} and e_{t2}, and

$$\forall\ m,\ (\texttt{fix}\ f(x).\, e_i,\ \texttt{fix}\ f(x).\, e_{t1},\ (\sigma_1,\ \emptyset,\ m)) \in \mathcal{V}(\!|\varphi\tau_1 \xrightarrow{\mathbb{C}(\sigma\kappa')} \varphi\tau_2|\!) \tag{A5.2}$$

  and

$$\forall\ m,\ (\texttt{fix}\ f(x).\, e_c,\ \texttt{fix}\ f(x).\, e_{t2},\ (\sigma_1 \uplus \sigma_1',\ \beta_1,\ m)) \in \mathcal{V}(\!|\varphi\tau_1|\!) \tag{A5.3}$$

  We can pick an arbitrary m and instantiate eq. (A5.2) with $m+1$ and then eq. (B6.2) (note that $\sigma_2 \sqsupseteq \sigma_1$, $m < m+1$) in order to derive

$$\forall\ m,\ (e_i[x/v_i'][f/\texttt{fix}\ f(x).\, e_i],\ e_{t1}[x/v_t'][f/\texttt{fix}\ f(x).\, e_{t1}],\ (\sigma_2,\ \emptyset,\ m)) \in \mathcal{E}(\!|\varphi\tau_2|\!)^{\varphi\kappa'} \tag{C}$$

  We can pick an arbitrary m and instantiate eq. (A5.3) with $m+1$ and then eq. (B6.3) (note that $(\sigma_2 \uplus \sigma_2',\ \beta_2,\ m) > (\sigma_1 \uplus \sigma_1',\ \beta_1,\ m+1)$) in order to derive

$$\forall\ m,\ (e_c'[x/v_c'][f/\texttt{fix}\ f(x).\, e_c'],\ e_{t2}'[x/v_t'][f/\texttt{fix}\ f(x).\, e_{t2}'],\ (\sigma_2' \uplus \sigma_2,\ \beta_2,\ m)) \in \mathcal{E}(\!|\varphi\tau_2|\!)^{\varphi\kappa'}$$

  Let $l_n' = \texttt{fresh}_L(\sigma_2')$. Using Lemma 36 and the above equation we derive

$$\forall\ m,\ (e_c'[x/v_c'][f/\texttt{fix}\ f(x).\, e_c'],\ e_{t2}'[x/v_t'][f/\texttt{fix}\ f(x).\, e_{t2}'],\ (\sigma_2 \uplus \sigma_2'[l_n' \mapsto \square],\ \beta_2,\ m)) \in \mathcal{E}(\!|\varphi\tau_2|\!)^{\varphi\kappa'} \tag{D}$$

  We now instantiate Lemma 41 with eq. (D) and we derive that

$$e_c[x/v_c'][f/\texttt{fix}\ f(x).\, e_c] \Downarrow v_c \tag{D1}$$

$$e_{t2}'[x/v_t'][f/\texttt{fix}\ f(x).\, e_{t2}'],\ \sigma_2 \uplus \sigma_2'[l_n' \mapsto \square],\ \_ \Downarrow^{\mathbb{C}(l_n')}_{L_2,\ \beta_2} l_2,\ \sigma_2 \uplus \sigma_3',\ \_,\ c_3$$

  and consequently using Lemma 29

$$(f\ v_t')[f/\texttt{fix}\ f(x).\, e_{t2}'],\ \sigma_3 \uplus \sigma_2'[l_n' \mapsto \square],\ \_ \Downarrow^{\mathbb{C}(l_n')}_{L_2,\ \beta_2} v_2,\ \sigma_3 \uplus \sigma_3',\ \_,\ c_3 + 1 \tag{D2}$$

  We also derive

$$c_3' \leq \varphi\kappa' \tag{D3}$$

  and

$$\forall\ m,\ (v_c,\ l_n',\ (\sigma_3 \uplus \sigma_3'[l_n' \mapsto v_2],\ \beta_2,\ m)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{D4}$$

We consider the following cases.

– $r = \mathbb{S}$

Let $l_n = \texttt{fresh}_L(\sigma_2)$. Using Lemma 36 and eq. (C) we can derive that

$$\forall\ m,\ (e_i[x/v_i'][f/\texttt{fix}\ f(x).\, e_i],\ e_{t1}[x/v_t'][f/\texttt{fix}\ f(x).\, e_{t1}],\ (\sigma_2[l_n \mapsto \square],\ \emptyset,\ m)) \in \mathcal{E}(\!|\varphi\tau_2|\!) \tag{C.1}$$

We now instantiate Lemma 41 with eq. (C.1) and we derive that

$$e_{t1}'[x/v_t'][f/\texttt{fix}\ f(x).\, e_t'][e_{t_1},\ \sigma_2[l_n \mapsto \square],\ t_2 + 1 \Downarrow^{\mathbb{C}(l_n)}_{L_1,\ \emptyset} v_1,\ \sigma_3,\ t_3,\ \_$$

and consequently

$$(f\ v_t')[f/\texttt{fix}\ f(x).\, e_{t1}'],\ \sigma_2[l_n \mapsto \square],\ t_2 + 1 \Downarrow^{\mathbb{C}(l_n)}_{L_1,\ \emptyset} v_1,\ \sigma_3,\ t_3,\ \_ \tag{C1}$$

We also derive using the fact that the evaluation relation in the source is deterministic

$$\forall\ m,\ (v_i,\ l_n,\ (\sigma_3[l_n \mapsto (v_1,\ [])],\ \emptyset,\ m)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{C2}$$

We can now show the goals.

1. From eq. (A1), eq. (B1) and eq. (D1) we can derive that

$$\theta_c(e_1\ e_2) \Downarrow v_c$$

2. From eq. (A2), eq. (B2) and eq. (C1) we can derive that

$$\theta_t{}^{\ulcorner} e^{\urcorner},\ \sigma_i,\ t_0 \Downarrow^{\mathbb{S}}_{L_1,\emptyset} l_n,\ \sigma_3[l_f \mapsto (\texttt{fix}\, f(x).\, e_{t1},\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3))][l_n \mapsto v_1],\ t_3{+}1,\ \_$$

Furthermore, we will show that $\texttt{inv}(\sigma_3[l_f \mapsto (\texttt{fix}\, f(x).\, e_{t1},\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3))][l_n \mapsto v_1],\ \beta_o,\ l_n)$

Proof. Let $l \in \hat{\mathcal{R}}_{(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ \texttt{dom}(bij_o))}(\texttt{FL}(l_n))$. We know that $l_f \in \texttt{dom}(\beta_1)$ and consequently from Lemma 30 we can derive that $l_f \in \mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o))$ or $l_f \in \texttt{dom}(b_o)$. Thus, $\texttt{path}(\texttt{dom}(\beta),\ \texttt{edges}(\sigma_1),\ l_f)$. Using Lemma 25 we can derive that $\texttt{path}(\texttt{dom}(\beta_o),\ \texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ l_n)$, and thus, by the definition of $\hat{\mathcal{R}}_{(\cdot,\ \cdot)}(\cdot)$, that $l = l_n$. Also, using Lemma 29, we derive that there is no $(l_s',\ l_d',\ f',\ t_1',\ t_2') \in \texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])])$ such that $\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ l_s')$ and $t_1' < t_1$, $t_2 < t_2'$. We conclude that $l_n \in \texttt{trg}(\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \texttt{dom}(\beta_o)))$ □

3. We can use eq. (D2) and the definition of change propagation to derive that

$$[(l_f,\ l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3)],\ \sigma_3,\ \sigma_2',\ \beta_2 \rightsquigarrow \sigma_3'[l_n' \mapsto (v_2,\ [])],\ \beta_2[l_n \mapsto l_n'],\ c_3 + 2$$

We will also show that $\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_{t1},\ (l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3) :: \vec{e})][l_n \mapsto (v,\ [])]),\ \beta_o) = \mathcal{D}(\texttt{edges}(\sigma_2),\ \beta_o) +\!\!+ [(l_f,\ l_n,\ \lambda f.\, f\ v_t',\ t_2,\ t_3)]$

Proof. Note that

$$\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \beta_o) =$$

$$\mathcal{D}(\texttt{edges}(\sigma_2) \cup (\texttt{edges}(\sigma_3) \setminus \texttt{edges}(\sigma_2)) \cup (l_f,\ l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3),\ \beta_o)$$

We know that $l_f \in \texttt{dom}(\beta_1)$ and consequently from Lemma 30 we can derive that $l_f \in \mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o))$ and $\texttt{path}(\texttt{dom}(\beta),\ \texttt{edges}(\sigma_1),\ l_f)$. We can use Lemma 27 (all the other preconditions follow from Lemma 29) to derive the result. □

From the above, Lemma 33 and Lemma 32 we derive

$$\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \beta_o),\ \sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ \sigma_o,\ \beta_o$$
$$\rightsquigarrow$$
$$\sigma'_3[l'_n \mapsto (v_2,\ [])],\ \beta_2[l_n \mapsto l'_n],\ c_2 + c_3 + 2$$

4. From eq. (A5), eq. (B5) and eq. (D3) we can derive that

$$c_2 + c_3 + 2 - c' \leq \varphi(\kappa_1 + \kappa_2 + \kappa' + c_{app}(\epsilon, \mathbb{C}))$$

5. Using the fact that $\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3) :: \vec{e})] \sqsupseteq \sigma_3$ we derive $\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3) :: \vec{e})][l_n \mapsto (v,\ [])] \sqsupseteq \sigma_3[l_n \mapsto (v,\ [])]$. From eq. (C2), eq. (D4), Lemma 36 and the definition of the relation we derive that

$$(v_i,\ v_c,\ l_n,\ (\sigma_3[l_f \mapsto \ldots][l'_n \mapsto (v_1,\ [])],\ \sigma'_3[l'_n \mapsto (v_2,\ [])],\ \beta_2[l_n \mapsto l'_n],\ m - j_i))$$
$$\in \mathcal{V}[\![\varphi\tau_2]\!]$$

– $m = \mathbb{C}(l_n)$

We now instantiate Lemma 41 with eq. (C.1) and we derive that

$$e_{t1}[x/v'_t][f/\texttt{fix}\ f(x).\, e_{t1}]e_{t_1},\ \sigma_2,\ t_2 + 1 \Downarrow^{\mathbb{C}(l_n)}_{L_1,\ \emptyset} v_1,\ \sigma_3,\ t_3,\ \_$$

and thus, we obtain

$$(f\ v'_t)[f/\texttt{fix}\ f(x).\, e_{t1}],\ \sigma_2,\ t_2 + 1 \Downarrow^{\mathbb{C}(l_n)}_{L_1,\ \emptyset} v_1,\ \sigma_3,\ t_3,\ \_ \tag{C1}$$

and

$$\forall\ m,\ (v_i,\ l_n,\ (\sigma_3[l_n \mapsto v_1],\ \emptyset,\ m)) \in \mathcal{V}(\!|\varphi\tau_2|\!) \tag{C2}$$

We can now show the goals.

1. From eq. (A1), eq. (B1) and eq. (D1) we can derive that

$$\theta_c(e_1\ e_2) \Downarrow v_c$$

2. From eq. (A2), eq. (B2) and eq. (C1) we can derive that

$$\theta_t{}^{\ulcorner} e^{\urcorner},\ \sigma_i,\ t_0 \Downarrow^{\mathbb{C}(l_n)}_{L_1,\ \emptyset} v_1,\ \sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_{t1},\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3))],\ t_3 + 1,\ \_$$

Furthermore, we will show that $\texttt{inv}(\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_{t1},\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3))][l_n \mapsto v_1],\ \beta_o,\ l_n)$

Proof. Let $l \in \hat{\mathcal{R}}_{(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ \texttt{dom}(\beta_o))}(\texttt{FL}(l_n))$. We know that $l_f \in \texttt{dom}(\beta_1)$ and consequently from Lemma 30 we can derive that $l_f \in \mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o))$ or $l_f \in \texttt{dom}(b_o)$. Thus, $\texttt{path}(\texttt{dom}(\beta),\ \texttt{edges}(\sigma_1),\ l_f)$. Using Lemma 25 we can derive that $\texttt{path}(\texttt{dom}(\beta_o),\ \texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ l_n)$, and thus, by the definition of $\hat{\mathcal{R}}_{(\cdot,\ \cdot)}(\cdot)$, that $l = l_n$. Also, using Lemma 29, we derive that there is no $(l'_s,\ l'_d,\ f',\ t'_1,\ t'_2) \in \texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])])$ such that $\texttt{path}(\texttt{dom}(\beta_o),\ \sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])],\ l'_s)$ and $t'_1 < t_1,\ t_2 < t'_2$. We conclude that $l_n \in \texttt{trg}(\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \texttt{dom}(\beta_o)))$ □

3. We can use eq. (D2) and the definition of change propagation to derive that

$$[(l_f,\ l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3)],\ \sigma_3,\ \sigma'_2,\ \beta_2 \rightsquigarrow \sigma'_3[l'_n \mapsto (v_2,\ [])],\ \beta_2[l_n \mapsto l'_n],\ c_3 + 2$$

We will also show that $\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_{t1},\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3) :: \vec{e})]),\ \beta_o) = \mathcal{D}(\texttt{edges}(\sigma_2),\ \beta_o) +\!\!+ [(l_f,\ l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3)]$

Proof. Note that

$$\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v,\ [])]),\ \beta_o) =$$

$$\mathcal{D}(\texttt{edges}(\sigma_2) \cup (\texttt{edges}(\sigma_3) \setminus \texttt{edges}(\sigma_2)) \cup (l_f,\ l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3),\ \beta_o)$$

We know that $l_f \in \texttt{dom}(\beta_1)$ and consequently from Lemma 30 we can derive that $l_f \in \mathcal{D}(\texttt{edges}(\sigma_1),\ \texttt{dom}(\beta_o))$ and $\texttt{path}(\texttt{dom}(\beta),\ \texttt{edges}(\sigma_1),\ l_f)$. We can use Lemma 27 (all the other preconditions follow from Lemma 29) to derive the result. □

From the above, Lemma 33 and Lemma 32 we derive

$$\mathcal{D}(\texttt{edges}(\sigma_3[l_f \mapsto \ldots]),\ \beta_o),\ \sigma_3[l_f \mapsto \ldots],\ \sigma_o,\ \beta_o$$
$$\rightsquigarrow$$
$$\sigma'_3[l'_n \mapsto (v_2,\ [])],\ \beta_2[l_n \mapsto l'_n],\ c_2 + c_3 + 2$$

4. From eq. (A5), eq. (B5) and eq. (D3) we can derive that

$$c_2 + c_3 + 2 - c' \leq \varphi(\kappa_1 + \kappa_2 + \kappa' + c_{app}(\epsilon, \mathbb{C}))$$

5. Using the fact that $\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3) :: \vec{e})] \sqsupseteq \sigma_3$ we derive $\sigma_3[l_f \mapsto (\texttt{fix}\ f(x).\, e_t,\ (l_n,\ \lambda f.\, f\ v'_t,\ t_2,\ t_3) :: \vec{e})][l_n \mapsto (v,\ [])] \sqsupseteq \sigma_3[l_n \mapsto (v,\ [])]$. From eq. (C2), eq. (D4), Lemma 36 and the definition of the relation we derive that

$$(v_i,\ v_c,\ l_n,\ (\sigma_3[l_f \mapsto \ldots][l_n \mapsto (v_1,\ [])],\ \sigma'_3[l'_n \mapsto (v_2,\ [])],\ \beta_2[l_n \mapsto l'_n],\ m - j_i)) \in \mathcal{V}[\![\varphi\tau_2]\!]$$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_\epsilon e : \tau \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \forall x \in \Gamma,\ \ \Delta; \Phi \models \Gamma(x) \sqsubseteq \square(\Gamma(x)) \qquad \kappa = (\epsilon = \mathbb{C}\ ?\ \kappa' :\ 0)}{\Delta; \Phi; \Gamma, \Gamma' \vdash_\epsilon e : \square(\tau) \mid \kappa \hookrightarrow \ulcorner e \urcorner}$ **nochange**

From the induction hypothesis we derive that $(\theta_i e,\ \theta_c e,\ \theta_t \ulcorner e \urcorner,\ W) \in \mathcal{G}[\![\tau]\!]\kappa$. We know that $(\theta_i,\ \theta_c,\ \theta_t,\ W) \in \mathcal{G}[\![\phi\Gamma;\ \phi\Gamma']\!]$ We can derive that there are valuations θ'_i, θ'_t and θ'_c such that

$$(\theta'_i,\ \theta'_c,\ \theta'_t,\ W) \in \mathcal{G}[\![\phi\Gamma]\!] \tag{2}$$

Using Theorem 43 we can also derive that (note that $W = (\sigma_i,\ \sigma_c,\ \beta,\ m)$

$$(\theta'_i,\ \theta'_c,\ \theta'_t,\ (\sigma_i,\ \sigma_c,\ \beta,\ m)) \in \mathcal{G}[\![\square(\phi\Gamma)]\!] \tag{3}$$

From the above, since $\texttt{FL}(\theta_t \ulcorner e \urcorner) = \texttt{FL}(\theta'_t \ulcorner e \urcorner) = \texttt{FL}(\ulcorner e \urcorner)$ we derive $\mathcal{R}_{\sigma_i}(\texttt{FL}(\theta_t \ulcorner e \urcorner)) \cup \texttt{dom}(\beta) = 0$. We can now use Lemma 39 to derive that $(\theta_i e,\ \theta_c e,\ \theta_t \ulcorner e \urcorner,\ W) \in \mathcal{G}[\![\square\tau]\!]0$.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa' \hookrightarrow \ulcorner e \urcorner \qquad \Delta; \Phi \wedge C \models \kappa' \leq \kappa \qquad \forall x \in \Gamma\ \ \Delta; \Phi \wedge \neg C \models \Gamma(x) \sqsubseteq \square(\Gamma(x))}{\Delta; \Phi; \Gamma \vdash_{\mathbb{S}} e : \tau \mid \kappa \hookrightarrow \ulcorner e \urcorner}$ **split**

We consider the following cases.

- $\models \phi C$
  Then we can derive that $\models \phi\kappa' \leq \phi\kappa$. From the induction hypothesis we derive that

$$(\theta_i e,\ \theta_c e,\ \theta_t e,\ W) \in \mathcal{E}[\![\tau]\!]^{\kappa'}$$

  From lemma Theorem 43 we show that

$$(\theta_i e,\ \theta_c e,\ \theta_t e,\ W) \in \mathcal{E}[\![\tau]\!]^{\kappa}$$

- $\not\models \phi C$

Using Theorem 43 we can derive that

$$(\theta_i,\ \theta_c,\ \theta_t,\ W) \in \mathcal{G}[\![\Box(\phi\Gamma)]\!]$$

We can subsequently show that $\mathcal{R}_{\sigma_i}(\texttt{FL}(\theta_t{}^{\ulcorner}e^{\urcorner})) \cap \texttt{dom}(\beta) = \emptyset$, since $\texttt{FL}(\theta_t{}^{\ulcorner}e^{\urcorner}) = \texttt{FL}(\theta_t)$ Then from Lemma 39 we can show that

$$(\theta_i e,\ \theta_c e,\ \theta_t e,\ W) \in \mathcal{E}[\![\tau]\!]^0$$

and consequently

$$(\theta_i e,\ \theta_c e,\ \theta_t e,\ W) \in \mathcal{E}[\![\tau]\!]^\kappa$$

Case $\dfrac{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau' \mid \kappa' \hookrightarrow {}^{\ulcorner}e^{\urcorner} \qquad \Delta;\Phi \models \tau' \sqsubseteq \tau \qquad \Delta;\Phi \models \kappa' \leq \kappa}{\Delta;\Phi;\Gamma \vdash_\epsilon e : \tau \mid \kappa \hookrightarrow {}^{\ulcorner}e^{\urcorner}}\ \sqsubseteq$

From the induction hypothesis we know that

$$(\theta_i e,\ \theta_c e,\ \theta_t{}^{\ulcorner}e^{\urcorner},\ W) \in \mathcal{E}[\![\tau']\!]^{\kappa'}$$

Using Theorem 43 we derive that

$$(\theta_i e,\ \theta_c e,\ \theta_t{}^{\ulcorner}e^{\urcorner},\ W) \in \mathcal{E}[\![\tau]\!]^{\kappa}$$

□